Modelling financial data with stochastic processes

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01.08.2012
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Outline

Introduction

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Aims of talk

- Give an overview on the most popular discrete time models for stock returns.
- Show that these models are able to capture most of the characteristics financial data exhibit.
- Show how to calibrate the models from historical data.
- Show how to price risk in this model.
- Show one possibility to price options within this framework.

The structure of the talk

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- First, we introduce some basic definitions. The content is based on the textbooks of [Brockwell and Davis, 1991] chapter 1, [White, 2001] chapter 3.
- Second, stochastic volatility models are introduced and their properties are investigated. This section is heavily based on [McNeil et al., 2005] chapter 4 and [Andersen et al., 2009].
- In the third part application to problems in quantitative finance are given. This part also relies on [McNeil et al., 2005] chapter 4 as well as a recent review paper of [Christoffersen et al., 2009] and references therein.

Stochastic processes

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Basic definitions

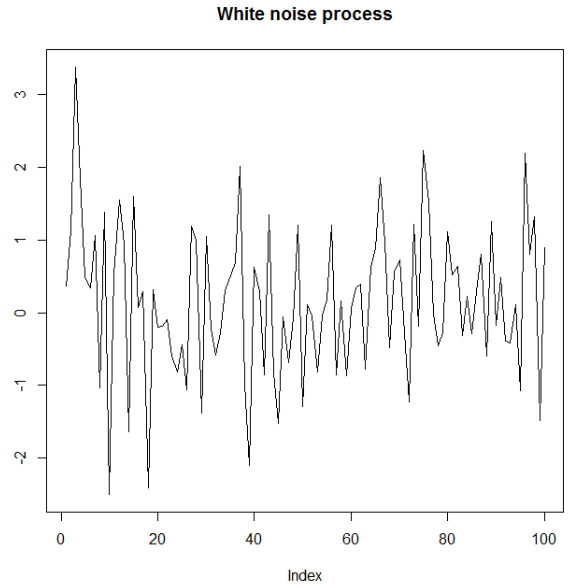
- In the following we consider a filtered probability space $(\Omega, \mathcal{A}, \mathfrak{F}_t, \mathcal{P})$.
- The index set $T = \mathbb{Z}$ or $T = \mathbb{N}$ will be interpreted as time points.

Definition

1. A stochastic process is a family of random variables $\{X_t, t \in T\}$ defined on $(\Omega, \mathcal{A}, \mathcal{P})$. We write $\mathbb{X} := (X_t)_{t \in T}$ for any stochastic process in discrete time.
2. The function $(X_.(\omega), \omega \in \Omega)$ on T are called realizations or sample paths of $\mathbb{X}$.

Examples I

White Noise: Let $\epsilon_t \overset{iid}{\sim} N(0,1)$, the process $X_t = \epsilon_t$ is called (strictly) white noise process $SWN(0,1)$ for short.

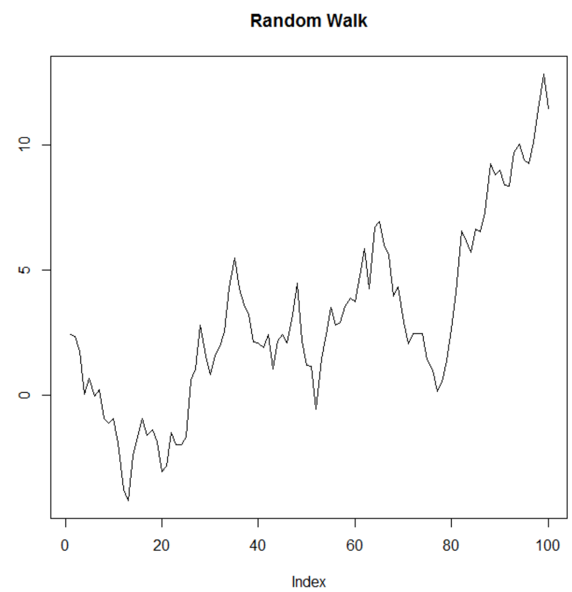


Examples II

Random Walk: Let $T = \mathbb{N}$ and ϵ_i be $SWN(0,1)$. The process

$$X_t = \sum_{i=1}^{t} \epsilon_i,$$

is called a random walk.

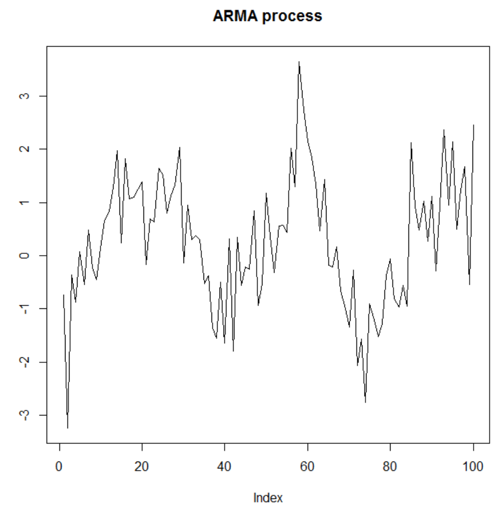


Examples III

Autoregressive moving average process (ARMA): Let $T = \mathbb{Z}$ and ϵ_i be $SWN(0, 1)$.

$$X_t = \mu + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

is called ARMA(p,q) process.



Basic properties

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Definition (The common distribution of a stochastic process)

Let $\mathcal{T} = \{\boldsymbol{t} = (t_1, ..., t_n)' \in T^n : t_1 < t_2 < ... < t_n, n = 1, 2, ...\}$. Then the finite dimensional distribution functions of $\mathbb{X}$ are defined by:

$$F_{\boldsymbol{t}}(\boldsymbol{x}) = P(X_{t_1} \leq x_1, ..., X_{t_n} \leq x_n), \quad \boldsymbol{x} = (x_1, ..., x_n)' \in \mathbb{R}^n.$$

Definition (The autocovariance function)

Let $Var(X_t) < \infty$ for all $t \in \mathbb{Z}$, then the autocovariance function is defined for all $s, t \in \mathbb{Z}$ by:

$$\gamma(s, t) = Cov(X_s, X_t) = E[(X_s - E[X_s])(X_t - E[X_t])].$$

Stationary I

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- With the assumption of stationary we reduce the number parameters that have to be estimated in order to describe the process completely.

Definition

A stochastic process $\mathbb{X}$ is called

1. *integrable*, if $E[|X_t|] < \infty$ for all $t \in T$,
2. *strictly stationary* or just *stationary*, if the joint distribution of $(X_{t_1}, \ldots, X_{t_k})'$ and $(X_{t_1+h}, \ldots, X_{t_k+h})'$ are the same for all $t_1, \ldots, t_k, h \in \mathbb{Z}$.
3. *weakly stationary* if for all $r, s, t \in T$:
   3.1 $E[|X_t|^2] < \infty$,
   3.2 $E[X_t] = \mu$,
   3.3 $\gamma(s, t) = \gamma(s + r, t + r)$.

Stationary II

- Note, if $\mathbb{X}$ is weakly stationary we have $\gamma(s,t) = \gamma(s-t,0) = \gamma(h)$, with $h = s - t \geq 0$.
- Also, we write $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$ for the autocorrelation function (ACF).
- A weakly stationary gaussian process is also strictly stationary.
- A strictly stationary process is also weakly stationary provided $Var(X_t) < \infty$.
- In general the converse is false.

Example I: The white noise process

- Let $\mathbb{X}$ be a SWN(0,1). This process is:
  1. integrable, since
  $$E[|X_t|] = \sqrt{2/\pi} < \infty,$$
  2. strictly stationary, as
  $$F_{(X_{t_1},\ldots,X_{t_k})}(\boldsymbol{x}) = \prod_{i=1}^{k} F_{X_{t_i}}(x_i) = \prod_{i=1}^{k} F_{X_{t_i+h}}(x_i),$$
  3. weakly stationary, as
     3.1 $E[|X_t|^2] = 1 < \infty,$
     3.2 $E[X_t] = 0$ for all t
     3.3 $\gamma(h) = \begin{cases} 1 & \text{for} \quad h = 0 \\ 0 & \text{for} \quad h \neq 0 \end{cases}$

Example II: The AR(1) process I

- Let ϵ_t be $SWN(0, 1)$. Consider the AR(1) process $X_t = \alpha X_{t-1} + \epsilon_t$. We may show the following: If $|\alpha| < 1$, then the AR(1) process is integrable, strictly stationary and weakly stationary.
- First, observe that

$$\begin{aligned} X_t &= \alpha(\alpha X_{t-2} + \epsilon_{t-1}) + \epsilon_t \\ &= \alpha^{k+1} X_{t-k-1} + \sum_{i=0}^{k} \alpha^i \epsilon_{t-i}. \end{aligned}$$

For $k \to \infty$ this yields to

$$X_t = \sum_{i=0}^{\infty} \alpha^i \epsilon_{t-i}.$$

Example II: The AR(1) process I

- Thus we have:
  1. $E[X_t] = 0$
  2. $Var(X_t) = \sum_{i=0}^{\infty} \alpha^{2i} Var(\epsilon_{t-i}) = 1/(1-\alpha^2)$
  3. and for $h > 0$

$$\begin{aligned}\gamma(h) &= E\left[\sum_{i=0}^{\infty} \alpha^i \epsilon_{t-h-i} \sum_{j=0}^{\infty} \alpha^j \epsilon_{t-j}\right] \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha^i \alpha^j E[\epsilon_{t-h-i}\epsilon_{t-j}] = \alpha^h \sum_{i=0}^{\infty} \alpha^{2i} \\ &= \alpha^h 1/(1-\alpha^2).\end{aligned}$$

  4. Especially we have $\rho(h) = \alpha^h$. Thus, the ACF is decaying exponentially fast for $|\alpha| < 1$.
- Since the AR(1) process is a weakly stationary gaussian process, it is also strictly stationary.

Ergodicity I

- Ergodicity of stochastic process is a crucial assumption when expected values or parameters are estimated.
- Under stationarity and ergodicity conditions a generalization of the strong law of large numbers is possible.

Ergodicity II

Definition (Measure preserving and ergodicity)

1. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space. The transformation $T : \Omega \to \Omega$ is measure preserving if it is measurable and if $P(T^{-1}A) = P(A)$ for all events $A \in \mathcal{A}$.
2. A stationary sequence $\mathbb{X}$ is ergodic if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} P(A \cap T^t B) = P(A)P(B),$$

for all events $A, B \in \mathcal{A}$ and for all measure preserving transformations T, s.t. $X_1(\omega) = X_1(\omega), X_2(\omega) = X_1(T\omega), \ldots, X_t(\omega) = X_1(T^{t-1}\omega)$.

- The random variables induced by measure preserving mappings are identically distributed, that is:

$$P(X_1 \leq x) = P(\{\omega : X_1(\omega) \leq x\}) = P(\{\omega : X_1(T\omega) \leq x\}) = P(X_2 \leq x).$$

The ergodic Theorem: A Law of large numbers

Theorem (Ergodic Theorem)

Let $\mathbb{X}$ be a stationary and ergodic sequence with $E|X_t| < \infty$. Then

$$\frac{1}{n}\sum_{t=1}^{n} X_t \overset{a.s.}{\to} E[X_t].$$

1. The classical (strong) law of large numbers is a special case: A sequence of iid. random variables $X_1, ...X_n$ is stationary and ergodic.
2. Note, that $Y_t = g(X_t)$ for some measurable map g is also stationary and ergodic. Provided that $\mathbb{X}$ is stationary and ergodic.
3. Thus, under ergodicity of the time series we can consistently estimate the moments like the expected value, variances or autocovariance based on realizations X_t $t = 0, ...T$ of the process $\mathbb{X}$.

The mixing property I

Definition

- The process $\mathbb{X}$ is said to be α or strong mixing if

$$\alpha_t = \sup\{|P(A \cap B) - P(A)P(B)| : A \in \mathfrak{F}^0_{-\infty}, B \in \mathfrak{F}_t^\infty\} \overset{t\to\infty}{\to} 0,$$

  where $\mathfrak{F}_a^b = \sigma(X_t, a \leq t \leq b)$.
- The process is $\mathbb{X}$ is said to be β-mixing or absolutely regular if

$$\beta_t = E[\sup_{B \in \mathfrak{F}_t^\infty} |P(B|\mathfrak{F}^0_{-\infty}) - P(B)|] \overset{t\to\infty}{\to} 0 \qquad (1)$$

The mixing property II

1. If $\mathbb{X}$ is absolutely regular it is also strong mixing.
2. If $\mathbb{X}$ is stationary and strong mixing, than the process is ergodic.
3. The mixing coefficient shows how fast the dependence decays over time.
4. For instance: if α_t decays exponentially fast, i.e. $\alpha_t = O(\rho^t)$, with $\rho \in (0, 1)$, we say that $\mathbb{X}$ is strongly mixing with geometrical decay.
5. The rate of α_t is closely related to the decay in the ACF.

Some examples

- Let X_t be SWN(0,1). This process is:
  1. ergodic, since ..., X_{t-2}, X_{t-1}, X_t, ... are independent and identically distributed.
  2. absolut regular since

     $$\beta_k = E[\sup |P(B|\mathfrak{F}_0) - P(B)|] = 0,$$

     because of the independence of all X_k from X_0 for $k \geq 1$.
  3. Consequently the iid. sequence is also strong mixing.
- Consider the stationary AR(1) process $X_t = \alpha X_{t-1} + \epsilon_t$, with $|\alpha| < 1$ and ϵ_t a white noise process.
  1. It can be shown, (see for instance [Mokkadem, 1988]) that the AR(1) process is absolutely regular with geometrical decay, whenever $|\alpha| < 1$
  2. Thus, the AR(1) process is also ergodic.

Volatility models

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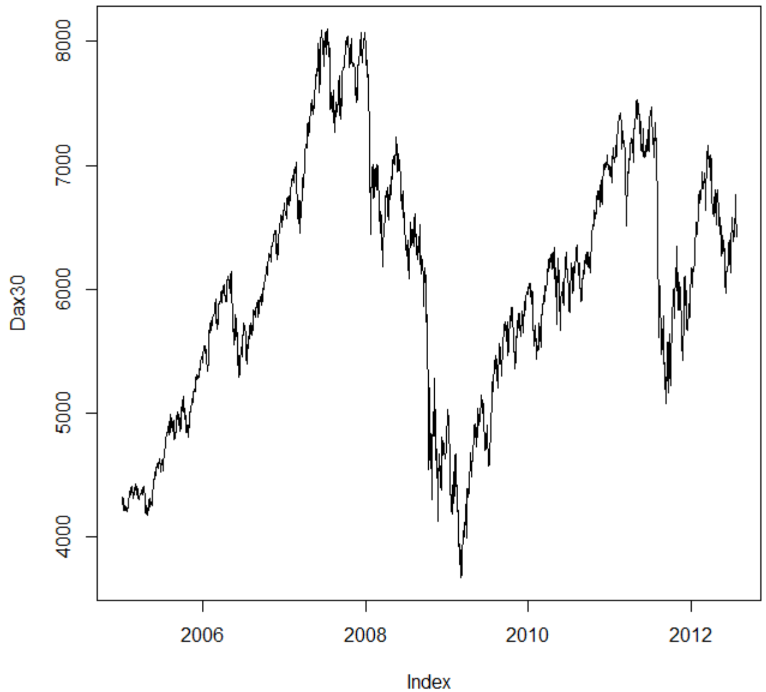
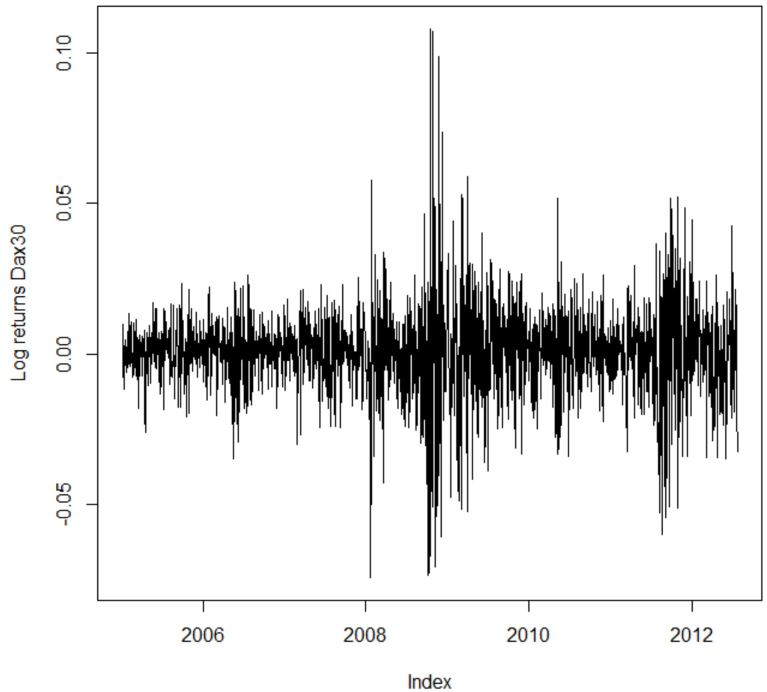
Asset returns and Volatility models I

- We observe the price S_t of an asset at time t as an realization of a stochastic process.
- Typically S_t is neither stationary nor ergodic, so we consider the transformation:
$$X_t = \log(S_t/S_{t-1}).$$
- The process X_t is said to be the process of the (log-)returns, which can be tested for stationarity.
- Compare both processes S_t and X_t:

Asset returns and Volatility models II



(a) Prices of DAX30



(b) Stock returns of DAX30

Stylized Facts of financial returns

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We would like to find functions(s) that model the following stylized facts (see [Rama, 2001])

- Absence of autocorrelations
- Slow decay of autocorrelation in absolute and squared returns
- Volatility clustering
- Heavy tails
- Conditional heavy tails
- Leverage Effect
- Gain / Loss asymmetry

Volatility models

- We consider the following volatility model in discrete time:

$$X_t = \mu_t + \sigma_t \epsilon_t, \tag{2}$$

  where
  1. $\mu_t := \mu(X_{t-1}, X_{t-2}, ..., \epsilon_{t-1}, \epsilon_{t-2}, ...)$ is a $\mathfrak{F}_{t-1}$ measurable function of past observations X_{t-i} and ”shocks” ϵ_{t-i} modeling the conditional mean of X_t,
  2. $\sigma_t := \sigma(X_{t-1}, X_{t-2}, ..., \epsilon_{t-1}, \epsilon_{t-2}, ...)$ is a $\mathfrak{F}_{t-1}$ measurable function modeling the conditional deviation of X_t often referred to as ”volatility” and
  3. ϵ_t is SWN(0,1).
- For instance take $\mu_t = \alpha_1 X_{t-1}, \sigma_t = \sigma$ for all $t \in \mathbb{Z}$, we have the AR(1) process with constant volatility σ, again.
- For simplicity we set $\mu_t \equiv 0$ in equation (2):

$$X_t = \sigma_t \epsilon_t. \tag{3}$$

- In the following we investigate the structure of processes of the form (3)

Definition (Linear GARCH(1,1))

A stochastic process $(X_t)_{t\in\mathbb{Z}}$ is called a **LGARCH(1,1)** process, if:

$$X_t = \sigma_t \epsilon_t, \quad (4)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \; t \in \mathbb{Z} \quad (5)$$

with

$$\boldsymbol{\theta} = (\alpha_0, \alpha_1, \beta_1) \in \Theta = \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+.$$

ϵ_t is $SWN(0,1)$.

- The linear GARCH model was introduced by [Bollerslev, 1986].
- In most application a simple LGARCH(1,1) model already gives a reasonable fit to financial data, see [Hansen and Lunde, 2005].

Basic properties of LGARCH(1,1) I

Theorem (Weak stationarity)

The LGARCH(1,1) process in (4) is a weakly stationary white noise process if and only if $\alpha_1 + \beta_1 < 1$ *with* $E[X_t] = 0$ *and* $Var(X_t) = \alpha_0/(1 - \alpha_1 - \beta_1)$.

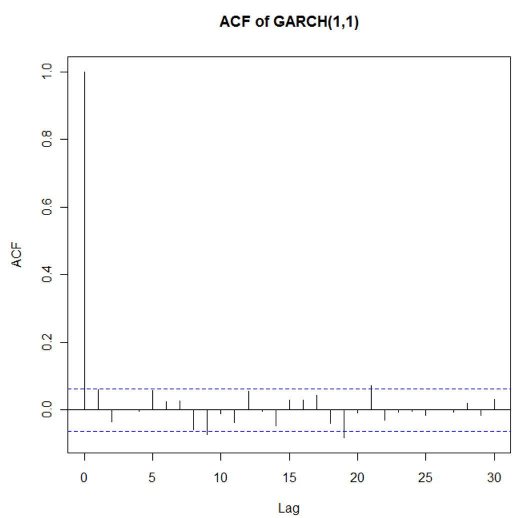
- When $\alpha_1 + \beta_1 = 1$ the LGARCH(1,1) process is not weakly stationary as $Var(X_t) = \infty$.
- Nevertheless, it can be shown that if $\alpha_1 + \beta_1 = 1$ the LGARCH(1,1) is strictly stationary.
- It can be shown that even if $\alpha_1 + \beta_1 = 1$ the LGARCH(1,1) is absolutely regular.

Theorem (Strict Stationarity and absolut regularity)

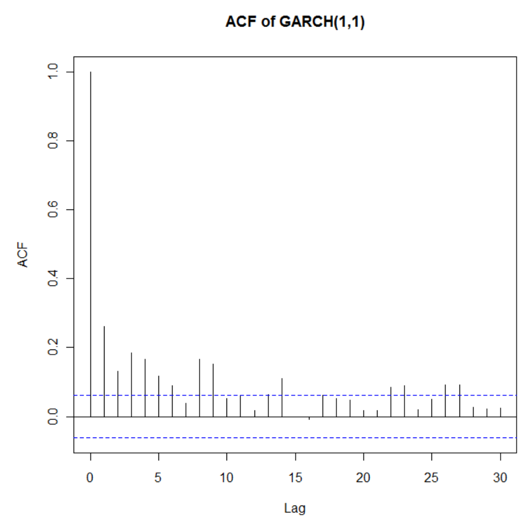
The LGARCH(1,1) process in (4) is strictly stationary and absolutely regular with exponential decay if $\alpha_1 + \beta_1 \leq 1$. *Hence, the process is also ergodic.*

Basic properties of LGARCH(1,1) II

- The proof for strict stationarity of the process is given in [Duan, 1997], the proof for absolute regularity of the process is given in [Francq and Zakoïan, 2006].
- The next pictures shows the autocorrelation of a LGARCH model with $\alpha_0 = 0.0001, \alpha_1 = 0.05, \beta_1 = 0.85$ and $\epsilon_t \overset{iid}{\sim} N(0,1)$.



(c) ACF of a LGARCH(1,1) process



(d) ACF of a squared LGARCH(1,1) process

The LGARCH(1,1) model and stylized facts

- It can be shown that the following stylized facts are captured by a simple LGARCH(1,1) process with gaussian innovations.
  1. Absence of autocorrelation, because a LGARCH(1,1) process is a martingale difference process, as $E[X_t|\mathfrak{F}_{t-1}] = 0$ for all $t \in \mathbb{Z}$.
  2. Volatility clustering because of autoregressive structure in X_t^2 rather X_t.
  3. Heavy tails, even if the conditional distribution is gaussian. This can be seen after some calculations:

$$E[X_t^4]/\sigma^4 = 3 + 6 \cdot \frac{\alpha_1^2}{1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2} > 3,$$

     whenever $1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2 > 0$ (which in application is often true, for instance $\alpha_1 = 0.05$ and $\beta_1 = 0.85$, then $1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2 = 0.185$.)
- The other stylized facts like leverage effects can be captured by various generalizations of the LGARCH model.
- For an overview we refer to part 1 of [Andersen et al., 2009].

Maximum-Likelihood estimation

- Suppose X has a density function $f_X(x;\boldsymbol{\theta})$ that depends on an unknown parameter(vector) $\boldsymbol{\theta}$, we wish to estimate.
- Based on an iid. sample $x_1, \ldots, x_n$ from X the Maximum-Likelihood estimator (MLE) for $\boldsymbol{\theta}$ is given by

$$\hat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}\in\Theta} L(\boldsymbol{\theta}; x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i, \boldsymbol{\theta}).$$

- It is convenient to maximize $LL = \ln L(\boldsymbol{\theta})$ instead of $L(\boldsymbol{\theta})$.
- Under suitable conditions, see for instance [Ferguson, 1996] chapter 18, we have:

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}_0) \xrightarrow{d} N(0, \Sigma),$$

where $\Sigma = -E[\nabla^2_{\boldsymbol{\theta}} \ln L(\boldsymbol{\theta})]^{-1}$.

LGARCH-Estimation I

- As the LGARCH(1,1) process is defined recursively the iid. assumption is violated.
- Based on the observed sample $x_1, \dots, x_n$ we can construct the joint density from:

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = f_{X_1}(x_1)\prod_{t=2}^{n} f_{X_t|X_{t-1},\dots,X_1}(x_t|x_{t-1},\dots,x_1).$$

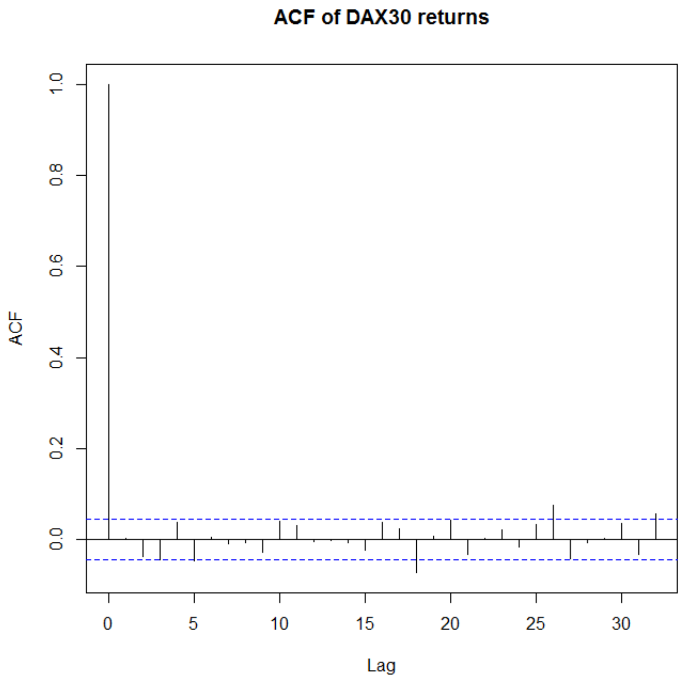
- Thus $LL(\boldsymbol{\theta}) = \ln f_{X_1}(x_1) + \sum_{t=2}^{n} \ln f_{X_t|X_{t-1},\dots,X_1}(x_t|x_{t-1},\dots,x_1)$.
- Consider the LGARCH(1,1) from equation 4, where $\epsilon_t \sim N(0,1)$. Given starting values (x_1, σ_1) MLE for $\boldsymbol{\theta} = (\alpha_0, \alpha_1, \beta_1)'$ is given after some calculations by:

$$LL(\boldsymbol{\theta}) = -c - \sum_{t=1}^{n} \log \sigma_t(\boldsymbol{\theta}) - 1/2 \sum_{t=1}^{n} x_t^2/\sigma_t^2(\boldsymbol{\theta}).$$

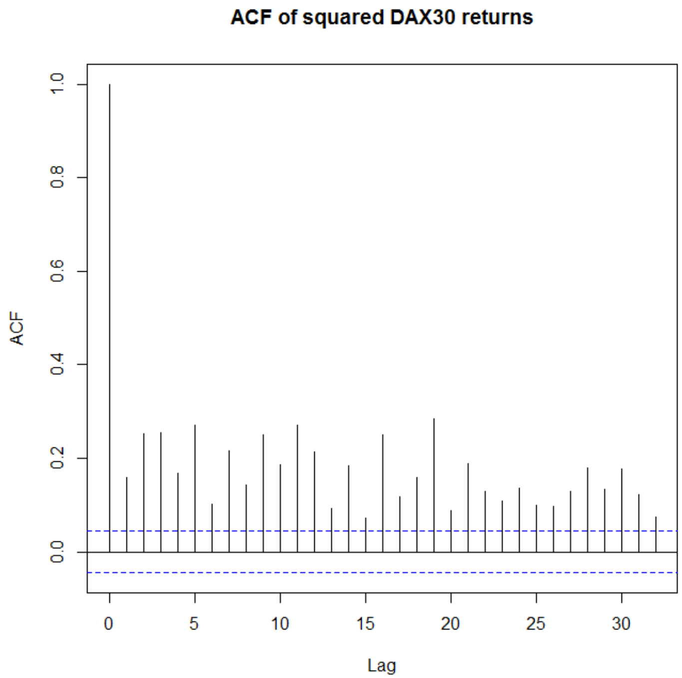
LGARCH-Estimation II

- The maximum $\hat{\boldsymbol{\theta}}_{ML}$ of $LL(\boldsymbol{\theta})$ is calculated using numerical methods.
- Under suitable conditions like ergodicity and strict stationarity the MLE is consistent and asymptotically normally distributed, even if the distribution of the residuals is unknown, see [Francq and Zakoïan, 2004], [Berkes et al., 2003] or [Mikosch and Straumann, 2006] resp.

Fitting the model to the DAX30 I



(e) ACF of DAX30 returns



(f) ACF of a squared DAX30 returns

Fitting the model to the DAX30 II

- From a first glance we can assume that the underlying process is a LGARCH(1,1) process.
- The estimated parameters of the LGARCH(1,1) process are:

$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	LL
1.39e-06	1.09e-01	8.89e-01	5733.608

- The parameters imply that the 4^{th} unconditional moment does not exist.
- A look at the residuals (and any test) rejects the hypothesis that the residuals are normal distributed but they are uncorrelated.

Fitting the model to the DAX30 III

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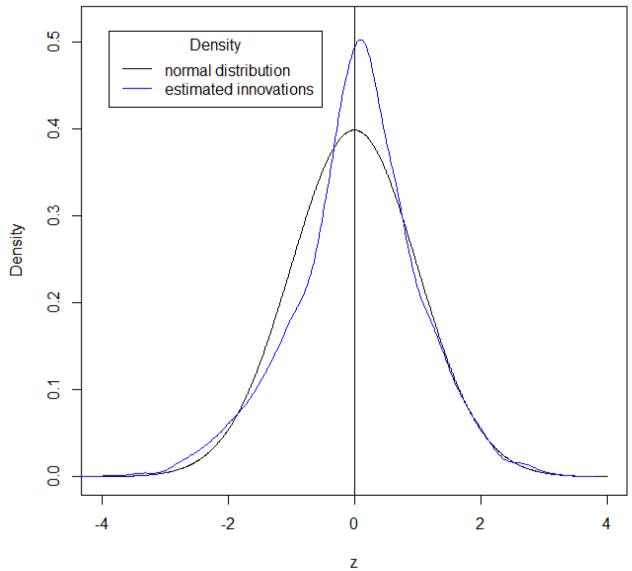
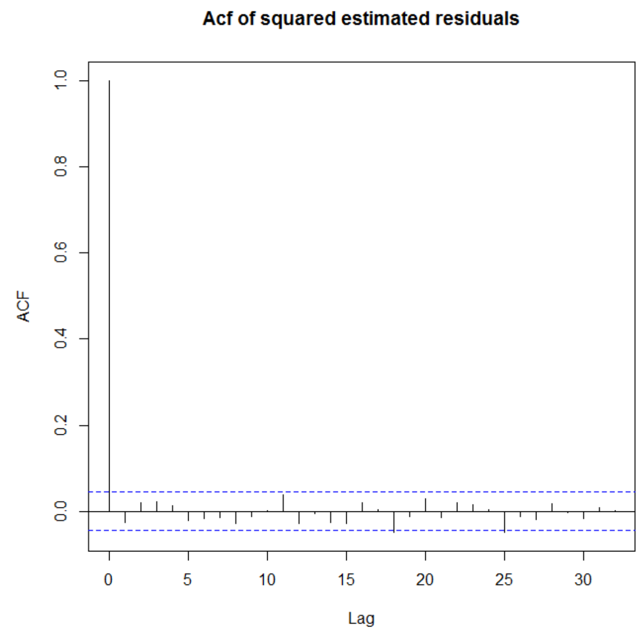


(g) Density estimates of the returns



(h) ACF of a squared estimated returns

Applications in finance

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The Value-at-risk I

- We want measure the risk of an investment, say an asset like an index fonds consisting of the DAX30.
- A prominent example of a risk measure is the so-called *Value-at-risk* (*VaR*).

Definition (Value-at-risk)

Given some confidence level $\alpha \in (0, 1)$. The *VaR* of a portfolio at given level α is the smallest number *l* s.t. the probability that the loss *L* exceeds *l* is not larger than $(1 - \alpha)$, i.e.

$$VaR_\alpha = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\},$$

where F_L is the loss distribution function.

1. The VaR_α is the quantile of the loss distribution.
2. The VaR_α is maximum loss that will not be exceeded with a given probability α.

The Value-at-risk II

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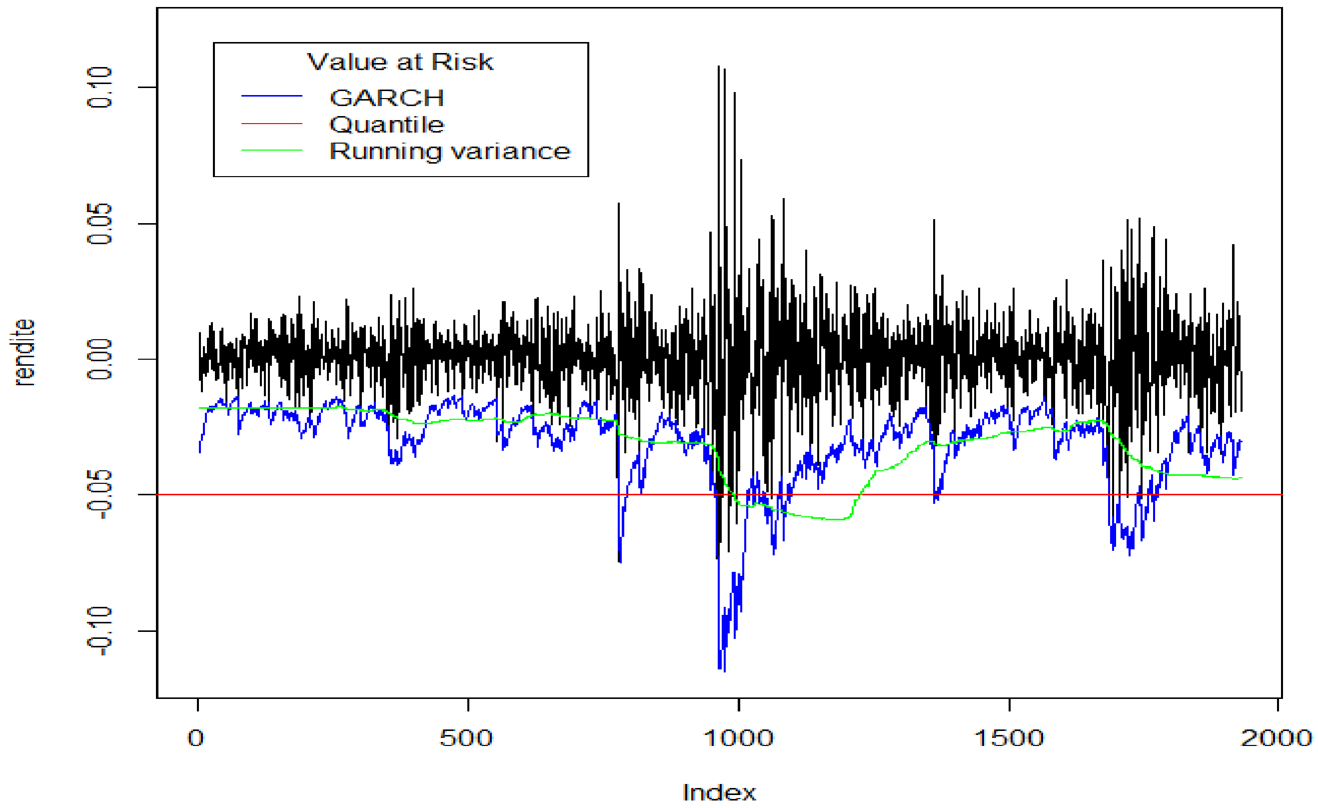
3. Suppose, the $L \sim N(0, \sigma^2)$, then $VaR_\alpha = \sigma \Phi^{-1}(\alpha)$ with $\Phi = N(0, 1)$.
4. Usually the loss distribution will be calculated for a given time horizon δ, for instance 1 day or 1 week ahead.
5. For the LGARCH(1,1) process in equation 4 the one-day-ahead forecast a time point t is then given by

$$VaR_\alpha^t = \hat{\sigma}_{t+1} \Phi^{-1}(\alpha),$$

   where σ_{t+1} is one-day forecast of the conditional volatility $\sigma_{t+1}^2 = \hat{\alpha}_0 + \hat{\alpha}_1 x_t^2 + \hat{\beta}_1 \sigma_t^2$.
6. As a comparison we use the α quantile and a non-parametric estimate that uses the past 250 observations to estimate the volatility at time t ('running variance').
7. We set $\alpha = 0.01$. In 1.91% the returns are lower than the predicted VaR. For the running variance this happens in 2.74% of the cases.

The Value-at-risk for the DAX30



Option Pricing

- A derivative security is a financial contract whose value is derived from an underlying.
- Broadly traded derivative securities are call or put options or futures.
- We focus on European call options.

Definition (Pay off function of a call option)

Denote the value of a call option at maturity T by $C_T(S_T, T, K)$, X_T the value of the underlying asset at time T and K the strike price. Then the pay-off function is given by:

$$C_T(S_T, T, K) := \max(0, S_T - K).$$

The current value of an option with maturity T is given by:

$$C_0(S_0, T, K) := e^{-rT} E_Q[\max(0, X_T - K)],$$

with r the interest rate and Q the risk neutral measure.

Deriving the risk neutral measure in LGARCH(1,1) models

Theorem

Let the log returns X_t follow a LGARCH(1,1) process under the physical/observed measure P.
Under the risk neutral measure Q the log returns have the following distribution

$$log(S_t/S_{t-1}) = r - \frac{h_t^2}{2} + \varepsilon_t^*,$$

where r is the risk free return, $\varepsilon_t^* = X_t + \sigma_t^2/2 - r$ is a risk neutral LGARCH process with:

$$\varepsilon_t^* | \mathcal{F}_{t-1} \sim N(0, h_t^2)$$

with risk-neutral variance dynamic:

$$h_t^2 = \alpha_0 + \alpha_1(\varepsilon_t^*) + \beta h_{t-1}^2.$$

Only for conditionally normal returns we have $h_t^2 = \sigma_t^2$

Simulating the Option price I

1. Q can not be obtained explicitly, thus the expected value in C_0 has to estimated via Monte-Carlo simulation
2. We simulate the T-day ahead Stock price under the risk neutral LGARCH process.
   2.1 Input: Stock price S_0, risk free interest r, strike price K, length of forecast period T, number of simulations m, initial conditional variance h_0^2, last observed risk free return ε_0^*.
   2.2 Simulate the paths of the stock price
3. Calculate the discounted mean of $max(S_T - K, 0)$.

Simulating the Option price II

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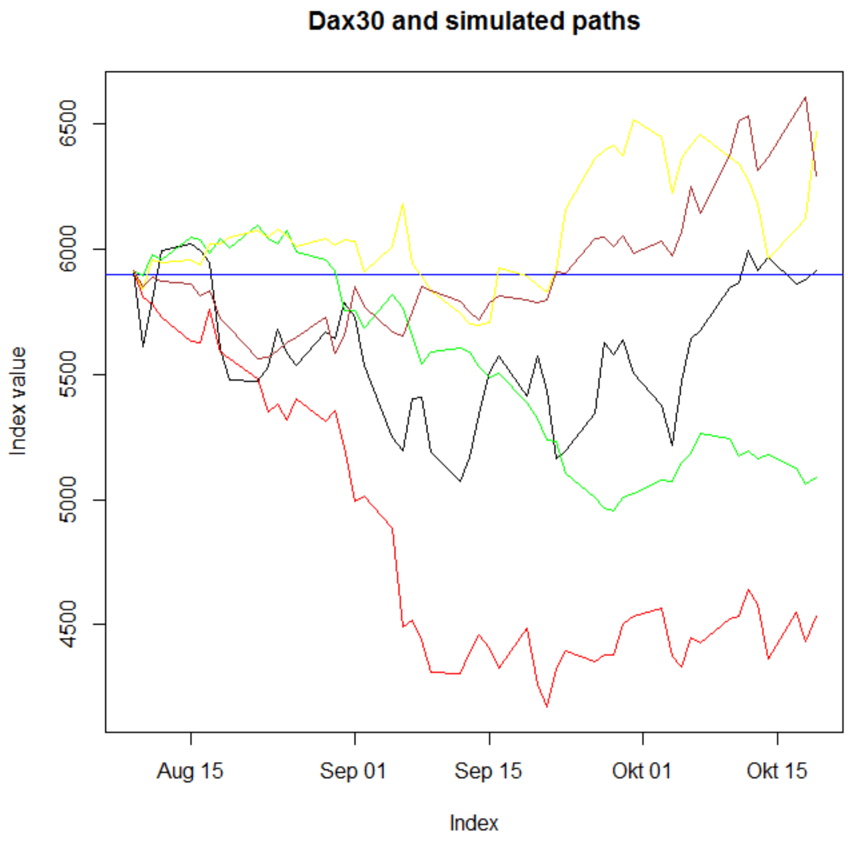

Option price for Dax30

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We want to calculate the option price for an Call option on the Dax30 that expires 52 days from the 09.08.2011 (20.10.2011). Closing stock price is 5917. As the risk free return we set $r = 1.2\%$ the one year EURIBOR at that time. As the observed price we use $C_{Bid}(X_T, T, K) - C_{Ask}(X_T, T, K))/2$. In order to simulate 100.000 paths we need 100 seconds.

Strike Price	Observed Price	Simulated Price	Black-Scholes
5450	6.415	7.178	4.173
5900	3.085	4.525	0.270
6050	2.205	3.776	0

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