M-estimators for augmented GARCH(1,1) processes

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Outline

Introduction

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Related works

1. Consistency and asymptotic normality of the Gaussian maximum likelihood estimator (QMLE) have been established for the linear GARCH (LGARCH) model by [Lee and Hansen, 1994] [Lumsdaine, 1996], [Berkes et al., 2003] and [Francq and Zakoïan, 2004].
2. Consistency and asymptotic normality in asymmetric GARCH (AGARCH) models have been shown by [Straumann and Mikosch, 2006], [Pan et al., 2008] and [Hamadeh and Zakoïan, 2011].
3. *M*-estimators in the linear GARCH model have been studied by [Muler and Yohai, 2008] and [Mukherjee, 2008].
4. Least absolute deviation estimators (LADE) have been proposed by [Peng and Yao, 2003] for the LGARCH and [Pan et al., 2008] for the threshold GARCH (TGARCH)

What is new in our work?

1. We establish consistency and asymptotic normality for the QMLE and the LADE for the class of augmented GARCH models that contains as special cases the LGARCH and TGARCH and many other models.
2. We apply this general framework to derive consistency and asymptotic normality of the QMLE and LADE in the LGARCH and the TGARCH model.
3. We investigate the finite sample properties of these estimators when the process is not weakly stationary.

Definition of M-estimators

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1. The basic idea is to describe estimators as functionals of the empirical distribution function.
2. Therefore, we write $\hat{\boldsymbol{\theta}}_n = T(F_n)$ for any estimator and let $K \subset \mathbb{R}^d$ be some compact set.
3. The M-estimator $\hat{\boldsymbol{\theta}}_n = T(X_1, \ldots, X_n)$ is defined as the solution of the following maximization problem

$$\hat{\boldsymbol{\theta}}_n := \arg\max_{\boldsymbol{\theta} \in K} \frac{1}{n} M_n(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta} \in K} \frac{1}{n} \sum_{i=1}^{n} \rho(X_i, \boldsymbol{\theta}). \tag{1}$$

4. The true parameter $\boldsymbol{\theta}_0$ is the solution of

$$\boldsymbol{\theta}_0 := \arg\max_{\boldsymbol{\theta} \in K} E_{\boldsymbol{\theta_0}}[\rho(X_i, \boldsymbol{\theta})]. \tag{2}$$

The augmented GARCH(1,1) model

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Definition and properties of augmented GARCH(1,1) processes

We follow [Hörmann, 2008] and define:

Definition

Let ϵ_t be an iid sequence and let $g(x)$ and $c(x)$ be real-valued and measurable functions. Assume that the stochastic recurrence equation

$$h_t := h(\sigma_t^2) = c(\epsilon_{t-1})h_{t-1} + g(\epsilon_{t-1}) \tag{3}$$

has a strictly stationary solution and assume $h_t : \mathbb{R}^+ \to \mathbb{R}^+$ is an invertible function. Then, an augmented GARCH(1,1) process $(X_t)_{t\in\mathbb{Z}}$ is defined by the equation

$$\begin{aligned} X_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= h_t^{-1}. \end{aligned} \tag{4}$$

Some Examples

Model	Specification
LGARCH(1,1) of [Bollerslev, 1986]	$\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta \sigma_{t-1}^2$
AGARCH(1,1) of [Ding et al., 1993]	$\sigma_t^2 = \omega + \alpha(\lvert X_{t-1}\rvert - \gamma X_{t-1})^2 + \beta \sigma_{t-1}^2$
NGARCH(1,1) of [Engle and Ng, 1993]	$\sigma_t^2 = \omega + \alpha(\epsilon_{t-1} - c)^2 \sigma_{t-1}^2 + \beta \sigma_{t-1}^2$
VGARCH(1,1) of [Engle and Ng, 1993]	$\sigma_t = \omega + \alpha(\epsilon_{t-1} - c)^2 + \beta \sigma_{t-1}^2$
TSGARCH(1,1) of [Schwert, 1989]	$\sigma_t = \omega + \alpha \lvert X_{t-1}\rvert + \beta \sigma_{t-1}$

Table: GARCH specification

Consistency and asymptotic normality of M-estimators in augmented GARCH models

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Consistency of M-estimators: Assumptions and Notation I

Assume that the following assumptions hold:

(P1) ϵ_t is an iid sequence,

(P2) $E[\log^+ |g_\theta(\epsilon_0)|] < \infty$, $E[\log^+ |c_\theta(\epsilon_0)|] < \infty$,

(P3) $-\infty < E[\log |c_\theta(\epsilon_0)|] < 0$.

1. Then, we can express the conditional volatility as

$$\sigma_t^2(\boldsymbol{\theta}) = h^{-1}\left(\sum_{i=1}^{\infty} g_\theta(\epsilon_{t-1-i}) \prod_{1\leq j<i} c_\theta(\epsilon_{t-i})\right).$$

2. Since we do not observe the whole process but only for $t \geq 1$, we work with the approximation:

$$\tilde{\sigma}_t^2(\boldsymbol{\theta}) = h^{-1}\left(\sum_{i=1}^{t-1} g_\theta(\epsilon_{t-1-i}) \prod_{1\leq j<i} c_\theta(\epsilon_{t-i})\right).$$

Consistency of M-estimators: Assumptions and Notation II

3. The QMLE is defined as

$$\hat{\boldsymbol{\theta}}_n^* = \arg\max_{\boldsymbol{\theta}\in K} -\frac{1}{n} M_n(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}\in K} -\frac{1}{2n}\sum_{t=1}^{n} \log \sigma_t^2(\boldsymbol{\theta}) + \frac{X_t^2}{\sigma_t^2(\boldsymbol{\theta})}.$$

4. The QMLE based on the approximation $\tilde{\sigma}_t$ is defined as

$$\hat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta}\in K} -\frac{1}{n} \tilde{M}_n(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}\in K} -\frac{1}{2n}\sum_{t=1}^{n} \log \tilde{\sigma}_t^2(\boldsymbol{\theta}) + \frac{X_t^2}{\tilde{\sigma}_t^2(\boldsymbol{\theta})}.$$

The following assumptions are needed to ensure the a.s. convergence of the approximation $\tilde{\sigma}_t$ to σ_t holds.

Consistency of M-estimators: Assumptions and Notation

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(F1) It holds that a.s. $c_{\boldsymbol{\theta}}(\epsilon_0) \geq 0$ and $g_{\boldsymbol{\theta}}(\epsilon_0) \geq 0$ for all $\boldsymbol{\theta} \in K$.

(F2) Additionally, we have $E[|c_{\boldsymbol{\theta}}(\epsilon_0)|^{\mu}] < 1$ and $E[|g_{\boldsymbol{\theta}}(\epsilon_0)|^{\mu}] < \infty$ for some $\mu > 0$ and for all $\boldsymbol{\theta} \in K$.

(F3) The function $h(\sigma_t^2(\boldsymbol{\theta})) = \sigma_t^{2\delta}(\boldsymbol{\theta})$ is continuous in $\boldsymbol{\theta} \in K$ for all t.

(F4) For every $\boldsymbol{\theta} \in K$ we have the following identifier condition: $\sigma_0^2(\boldsymbol{\theta}) = \sigma_0^2(\boldsymbol{\theta}_0)$ if and only if $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

Consistency of the QMLE for augmented GARCH models I

Theorem (Consistency)

Let $0 < \delta$ and let $\boldsymbol{\theta}_0 \in K$ for some compact set $K \subset \mathbb{R}^d$. Assume that (P1)-(P3) and (F1)-(F4) hold and that $E[\epsilon_0^2] = 1$. Then, it follows that a.s. $\hat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$.

1. The idea of the proof is to show in a first step that a.s. $\hat{\boldsymbol{\theta}}_n^* \to \boldsymbol{\theta}_0$.
2. In a second step we may demonstrate that a.s. $\hat{\boldsymbol{\theta}}_n \to \hat{\boldsymbol{\theta}}_n^*$.

Consistency of the LADE for augmented GARCH models I

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1. The LADE for the parameters of an augmented GARCH model can be defined by

$$\hat{\boldsymbol{\theta}}_n = \arg\max_{\boldsymbol{\theta}\in K} \frac{1}{n} M_n(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}\in K} -n^{-1} \sum_{t=1}^{n} |\log X_t^2 - \log \tilde{\sigma}_t{}^2(\boldsymbol{\theta})|. \tag{5}$$

2. $\hat{\boldsymbol{\theta}}_n^*$ is obtained replacing $\tilde{\sigma}_t{}^2$ by σ_t^2.

Theorem

Let $0 < \delta$ and $\boldsymbol{\theta}_0 \in K$ for some compact set $K \subset \mathbb{R}^d$. Assume (P1)-(P3) and (F1)-(F4) hold and let the iid sequence $v_t = \log \epsilon_t^2$ has unique median at 0. Then, it follows that a.s. $\hat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$.

Asymptotic normality of the QMLE I

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(N1) The true parameter $\boldsymbol{\theta}_0$ lies in the interior of K, denoted with $\mathring{K}$.

(N2) The distribution of ϵ_t is such that $E[\epsilon_0^2] = 1$ and $E[\epsilon_0^4] < \infty$.

(N3) There exists a convex set $\tilde{K} \subset K$, containing $\boldsymbol{\theta}_0$, such that $\sigma_t^2(\boldsymbol{\theta})$ is three times continuously differentiable in $\boldsymbol{\theta}$ with measurable derivatives such that

  (i) $E_{\boldsymbol{\theta}_0}[\log^+ \|\nabla_{\boldsymbol{\theta}} \sigma_t^2(\boldsymbol{\theta})\|_{\tilde{K}}] < \infty$,
  (ii) $E_{\boldsymbol{\theta}_0}[\log^+ \|\nabla_{\boldsymbol{\theta}}^2 \sigma_t^2(\boldsymbol{\theta})\|_{\tilde{K}}] < \infty$,
  (iii) $E_{\boldsymbol{\theta}_0}[\log^+ \|\nabla_{\boldsymbol{\theta}}^3 \sigma_t^2(\boldsymbol{\theta})\|_{\tilde{K}}] < \infty$ and

  In addition, the following moment conditions hold:

  (iv) $E_{\boldsymbol{\theta}_0}[\|\nabla_{\boldsymbol{\theta}} \psi_0(\boldsymbol{\theta})\|_{\tilde{K}}] < \infty$.
  (v) $E_{\boldsymbol{\theta}_0}\left[\left\|\frac{1}{\sigma_0^2(\boldsymbol{\theta}_0)} \nabla_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0)\right\|\right] < \infty$,
  (vi) $E_{\boldsymbol{\theta}_0}\left[\left\|\frac{1}{\sigma_0^2(\boldsymbol{\theta}_0)} \nabla_{\boldsymbol{\theta}}^2 \sigma_0^2(\boldsymbol{\theta}_0)\right\|\right] < \infty$,
  (vii) $E_{\boldsymbol{\theta}_0}\left[\left\|\frac{1}{\sigma_0^4(\boldsymbol{\theta}_0)} \nabla_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0) \nabla'_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0)\right\|\right] < \infty$,

(N4) The components of $\nabla_{\boldsymbol{\theta}} \sigma_t^2(\boldsymbol{\theta})$ are linearly independent random variables.

Asymptotic normality of the QMLE II

Theorem

Let $\delta > 0$ and let $\boldsymbol{\theta}_0 \in \mathring{K}$ for some compact set $K \subset \mathbb{R}^d$. Assume that (P1)-(P3), (F1)-(F4) and (N1)-(N4) hold. Then, it follows that

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} N(0, \boldsymbol{\Sigma}),$$

where

$$\boldsymbol{\Sigma} = (E[\epsilon_0^4] - 1) E_{\boldsymbol{\theta}_0} \left[\frac{1}{\sigma_0^4(\boldsymbol{\theta}_0)} \nabla_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0) \nabla_{\boldsymbol{\theta}}^{'} \sigma_0^2(\boldsymbol{\theta}_0) \right]^{-1}.$$

Asymptotic normality of the LADE I

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For the LADE we replace assumption (N2) by

(L2) $v_0 = \log \epsilon_0^2$ has a continuous distribution $G(v)$ with $G(0) = 1/2$ and density $g(0) > 0$ such that $\sup_v |g(v)| \leq B_1 < \infty$.

Theorem

Let $\delta > 0$ and $v_t = \log \epsilon_t^2$ have a unique median at 0 with density function $g(v)$ such that $g(0) > 0$. Assume that (P1)-(P3), (F1)-(F4) and (N1),(L2),(N3)-(N4) hold. Then, it follows that

$$\sqrt{n}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \xrightarrow{d} N(0, \Sigma), \tag{6}$$

where

$$\boldsymbol{\Sigma} = E_{\boldsymbol{\theta}_0}\left[\frac{1}{\sigma_0^4(\boldsymbol{\theta}_0)} \nabla_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0) \nabla'_{\boldsymbol{\theta}} \sigma_0^2(\boldsymbol{\theta}_0)\right]^{-1} / (4(g(0))^2).$$

Application to the LGARCH and TGARCH model

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Consistency and asymptotic normality of the QMLE in LGARCH(1,1) model I

1. Let $g(\epsilon_t) = \omega$, $c(\epsilon_t) = \alpha\epsilon_t^2 + \beta$ $h(x) = x$, we get the LGARCH(1,1) model of [Bollerslev, 1986]:
$$\sigma_t^2 = \omega + \alpha X_{t-1}^2 + \beta\sigma_{t-1}^2.$$
2. The parameter space is given by $K_c = [c, 1/c] \times \{\alpha \in \mathbb{R}_0^+, \beta \in [0, 1-c]\}$
3. Consistency and asymptotic normality of the QMLE and LADE follows by application of the previous theorems.
4. The assumptions are the same as in [Francq and Zakoïan, 2004].
5. The assumptions for the LADE are weaker than those in [Peng and Yao, 2003].

Consistency and asymptotic normality of the QMLE in TGARCH(1,1) model I

1. Let $g(\epsilon_t) = \omega$, $c(\epsilon_t) = \alpha^+|\epsilon_t| + \alpha^- \max(0, -\epsilon_t) + \beta$ $h(x) = x^{1/2}$, we get the TGARCH(1,1) model of [Zakoïan, 1994]:
$$\sigma_t = \omega + \alpha^+|X_{t-1}| + \alpha^- X_{t-1}^- + \beta\sigma_{t-1}.$$
2. The parameter space is given by
$$K_d = [d, 1/d] \times \{\alpha^+ \in [d, 1-d], \alpha^- \in [d, 1-d], \beta \in [0, 1-d]\}$$
3. Consistency and asymptotic normality of the QMLE and LADE follows by application of the previous theorems.
4. The assumptions for the QMLE are the same as in [Hamadeh and Zakoïan, 2011].
5. A similar result for a different type of LADE have been established by [Pan et al., 2008].

Finite sample properties

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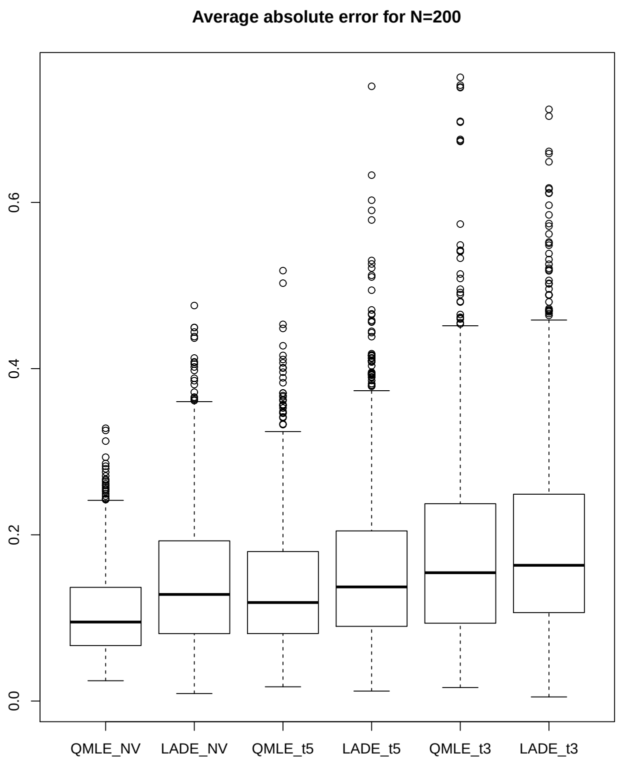
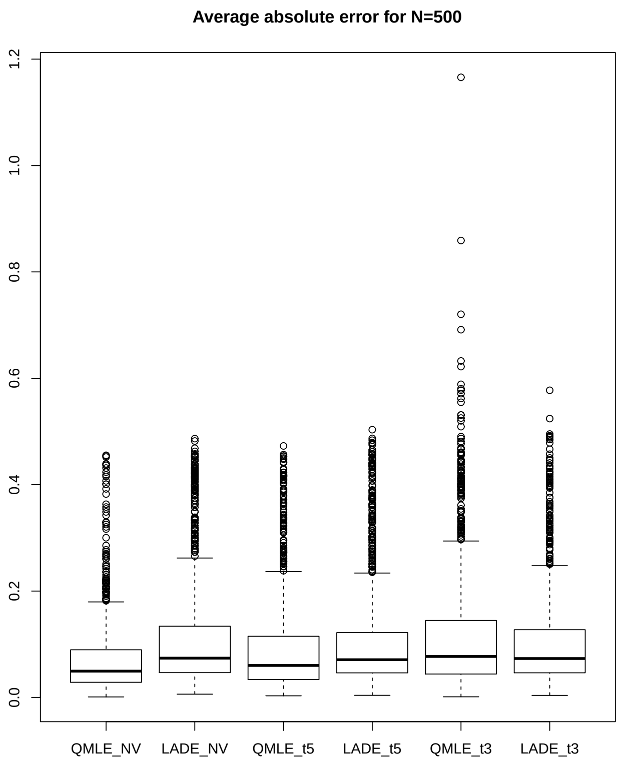
Simulation study

1. We simulate different GARCH models with different error distributions, namely $\epsilon_t \sim N(0,1), t(5), t(3)$.
2. The length of the time series are $T = 200, 500, 1000, 2000$.
3. We repeat the estimation of the parameters 1000 times for each time-length and error distribution and averaged the results.
4. We also report the $\text{RMSE}_j = \sqrt{1000^{-1} \sum_{i=1}^{1000} (\hat{\boldsymbol{\theta}}_{i,j} - \boldsymbol{\theta}_{0,j})^2}$ for the jth parameter.
5. The following boxplots report the absolute average error when an integrated LGARCH model is considered with
   - $\boldsymbol{\theta}_0 = (\omega_0, \alpha_0, \beta_0) = (0.1, 0.6, 0.4)$.

Results for the LGARCH(1,1)





Results for the LGARCH(1,1)

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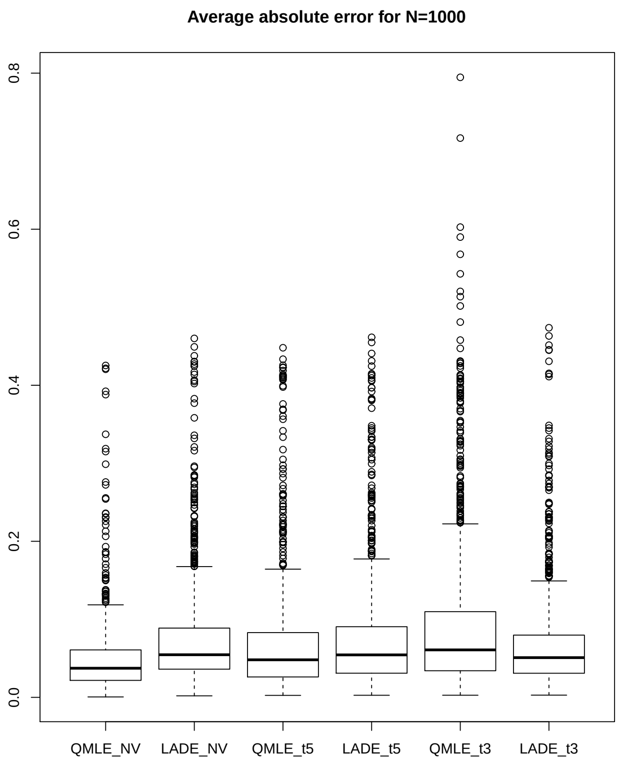

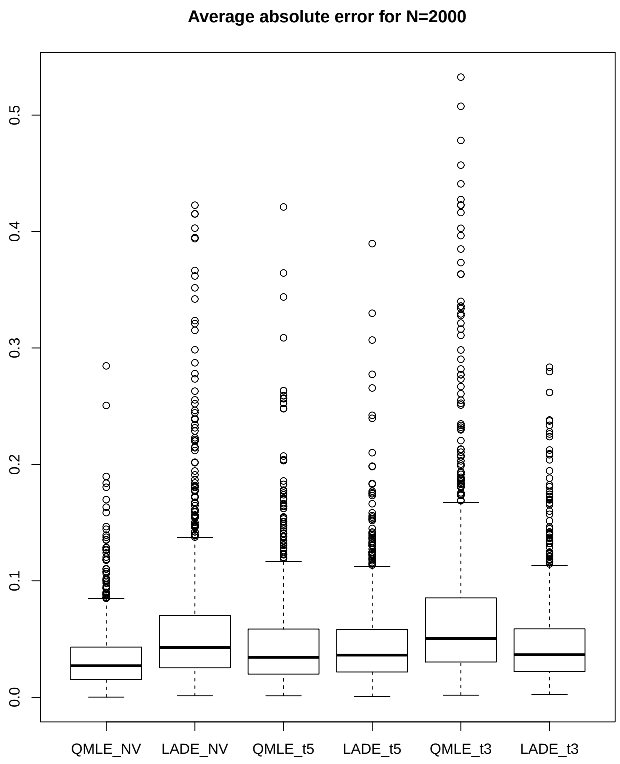

Conclusion

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Summary of the results

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1. We discuss the properties of the QMLE and the LADE for augmented GARCH models.
2. We derive conditions for consistency and asymptotic normality of the QMLE and the LADE that may easily checked whenever a specific model is considered.
3. Since we also propose a correction routine the results of the LADE are directly comparable with the QMLE.
4. The heavier the tails of the error distribution the more preferable is the LADE but the QMLE may still be useful for instance to obtain starting values for the LADE optimization routine.

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