MULTIVARIATE COPULA MODELS AT WORK: OUTPERFORMING THE "DESERT ISLAND COPULA"?

Matthias Fischer†, Christian Köck♮, Stephan Schlüter & Florian Weigert

† *Department of Computational Statistics*
Helmut-Schmidt University of Hamburg, Germany
Email: mfischer@hsu-hh.de

♮ *Department of Statistics and Econometrics*
University of Erlangen-Nürnberg, Germany
Email: Christian.Koeck@wiso.uni-erlangen.de

This research was kindly supported by the Hans Frisch-Stiftung.

SUMMARY

Since the pioneering work of Embrechts and co-authors in 1999, copula models enjoy steadily increasing popularity in finance. Whereas copulas are well-studied in the bivariate case, the higher-dimensional case still offers several open issues and it is by far not clear how to construct copulas which sufficiently capture the characteristics of financial returns. For this reason, elliptical copulas (i.e. Gaussian and Student-t copula) still dominate both empirical and practical applications. On the other hand, several attractive construction schemes appeared in the recent literature promising flexible but still manageable dependence models. The aim of this work is to empirically investigate whether these models are really capable to outperform its benchmark, i.e. the Student-t copula (which is termed by Paul Embrechts as "desert island copula" on account of its excellent fit to financial returns) and, in addition, to compare the fit of these different copula classes among themselves.

Keywords and phrases: KS-copula; Hierarchical Archimedian; Product copulas; Pair-copula decomposition

1 Introduction

The increasing linkages between countries, markets and companies require an accurate and realistic modelling of the underlying dependence structure. This applies to financial markets and, in particular, to the financial assets traded there-on. For a long time both practitioners and theorists rely on the multivariate normal (Gaussian) distribution as statistical fundament, seemingly ignoring that this model assigns too less probability mass to extremal events. In order to remove this drawback but still maintain many of the attractive properties, elliptical distributions (e.g. multivariate Student-t or multivariate generalized hyperbolic distribution) occasionally found its way into financial literature. Though being able to model heavy tails, elliptical distributions fail to capture asymmetric dependence structures. The copula concept, in contrast, which originally dates back to Sklar (1959) but was made popular to finance through the pioneering work of Embrechts and co-authors (1999) provides a flexible tool to capture different patterns of dependence. Within this work we assume that the reader is already familiar with the notion of copulas. Otherwise, we refer to Nelsen (2006) or Joe (1997). Whereas copulas are well-studied in the bivariate case, construction schemes for higher dimensional copulas are not. Recently, several publications on high-dimensional copulas appeared (e.g. Morillas, 2005, Palmitesta & Provasi, 2005, Savu & Trede, 2006, Liebscher, 2006, Aas et al., 2006). Each of them claims to provide a flexible dependence model, but there is no comprehensive comparison among these approaches, as far as we know. In particular, no references are found to the Student-t copula (i.e. the copula associated to the multivariate Student-t distribution) which is sometimes termed as "desert island copula" by Paul Embrechts on account of its excellent fit to multivariate financial return data.

The outline of this work is as follows: Section 2 overviews and connects several recent construction schemes of multivariate copulas. A short digression on goodness-of-fit measures can be found in section 3. Section 4 is dedicated to the description of the underling data sets, whereas the empirical results are summarized and discussed in section 5.

2 Constructing multivariate non-elliptical copulas

Among the classes of non-elliptical copulas, Archimedean copulas and its generalizations (section 2.1) enjoy great popularity. Above that, so-called pair-copula constructions are reviewed in section 2.2, where the joint distribution is decomposed into simple building blocks, so-called pair-copulas. Thirdly, we pick up the copulas associated to Köhler-Symanowski distributions in section 2.3 which have been successfully applied by Palmistesta & Provasi (2005) as models for financial returns. Finally, Liebscher's (2006) recent proposal to generalize given d-copulas is reviewed in section 2.4.

2.1 Multivariate Archimedean copulas

2.1.1 Classical multivariate Archimedean copulas

Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing and convex function with $\varphi(1) = 0$, $\varphi(0) \leq \infty$ and let $\varphi^{[-1]}$ be the so called pseudo-inverse of φ defined by

$$\varphi^{[-1]}(t) \equiv \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0), \\ 0 & \varphi(0) \leq t \leq \infty\,. \end{cases}$$

It can be shown (see, e.g. Nelsen, 2006) that

$$C(u_1, u_2) = \varphi^{[-1]}\Big(\varphi(u_1) + \varphi(u_2)\Big)$$

defines a class of bivariate copulas, the so-called Archimedean copulas. The function φ is called the (additive) generator of the copula. Furthermore, if $\varphi(0) = \infty$ the pseudo-inverse describes an ordinary inverse function (i.e. $\varphi^{[-1]} = \varphi^{-1}$) and in this case φ is known as a strict generator.

Given a strict generator $\varphi : [0,1] \to [0,\infty]$, bivariate Archimedean copulas can be extended to the d-dimensional case. For every $d \geq 2$ the function $C : [0,1]^d \to [0,1]$ defined as

$$C(\mathbf{u}) = \varphi^{-1}\Big(\varphi(u_1) + \varphi(u_2) + \cdots + \varphi(u_d)\Big) \tag{2.1}$$

is a d-dimensional Archimedean copula if and only if φ^{-1} is completely monotonic on $\mathbb{R}_+$, i.e. if $\varphi^{-1} \in \mathcal{L}_\infty$ with

$$\mathcal{L}_m \equiv \left\{ \phi : \mathbb{R}_+ \to [0,1] \,\Big|\, \phi(0) = 1,\, \phi(\infty) = 0,\, (-1)^k \phi^{(k)}(t) \geq 0 \,,\, k = 1, \ldots, m, \right\}.$$

The Gumbel copula derives from the generator $\varphi(t) = (-\ln t)^\theta, \theta \geq 1$ and the Clayton copula is generated by $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1), \theta > 0$. For an overview of further Archimedean copulas and the properties of the aforementioned ones, we refer the reader to the monographs by Nelson (2006) and Joe (1997).

2.1.2 Non-exchangeable Archimedean copulas

In order to increase flexibility and to allow for non-exchangeable dependence structures, several generalizations emerged in the recent literature: A simple one – the so-called fully nested Archimedean (FNA) copulas – can be found in Joe (1997, p. 89), Whelan (2004) and Savu & Trede (2006), and requires $d-1$ generator functions $\varphi_1, \ldots, \varphi_{d-1}$ with $\varphi_1^{-1}, \ldots, \varphi_{d-1}^{-1} \in \mathcal{L}_\infty$ and $\varphi_{i+1} \circ \varphi_i^{-1}(t) = \varphi_{i+1}(\varphi_i^{-1}(t)) \in \mathcal{L}^*_\infty$ for

$$\mathcal{L}^*_d = \left\{ \phi : \mathbb{R}_+ \to \mathbb{R}_+ \,\Big|\, \phi(0) = 0,\, \phi(\infty) = \infty,\, (-1)^{k-1} \phi^{(k)}(t) \geq 0 \,,\, k = 1, \ldots, d, \right\}.$$

The structure of FNA d-copulas is rather simple: One first couples u_1 and u_2. One then couples the copula of u_1 and u_2 with u_3 to a new copula which is coupled afterwards with u_4 and so on. Hence the FNA 4-copula is of the form

$$C(\mathbf{u}) = \varphi_3^{-1}\left[\varphi_3\left(\varphi_2^{-1}\left[\varphi_2\left(\varphi_1^{-1}\left[\varphi_1(u_1) + \varphi_1(u_2)\right]\right) + \varphi_2(u_3)\right]\right) + \varphi_3(u_4)\right]. \tag{2.2}$$

Figure 1 illustrates one possible FNA copula for dimension $d = 5$.

Secondly, mixing ordinary Archimedean and FNA copulas, partially nested Archimedean (PNA) copulas may be used. Again, for ease of notation, we focus on the 4-variate case

$$C(\mathbf{u}) = \varphi^{-1}\left[\varphi\left(\varphi_{12}^{-1}\left[\varphi_{12}(u_1) + \varphi_{12}(u_2)\right]\right) + \varphi\left(\varphi_{34}^{-1}\left[\varphi_{34}(u_3) + \varphi_{34}(u_4)\right]\right)\right]. \tag{2.3}$$

Note that $\varphi, \varphi_{12}, \varphi_{34}$ are generators with $\varphi^{-1}, \varphi_{12}^{-1}, \varphi_{34}^{-1} \in \mathcal{L}_\infty$ and $\varphi \circ \varphi_{12}^{-1}, \varphi \circ \varphi_{34}^{-1} \in \mathcal{L}^*_\infty$. Obviously, one first couples the pairs u_1, u_2 and u_3, u_4 with distinct generators. The resulting copula pair is then coupled using a third generator φ (which in turn might be coupled with

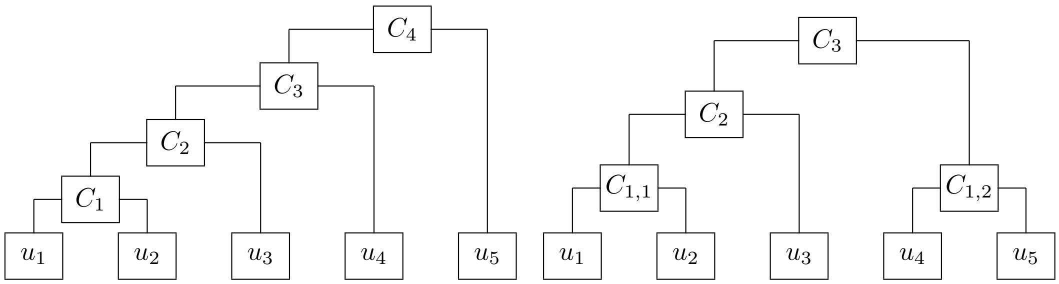

Figure 1: FNA copula (left) and PNA copula (right) for $d = 5$.

an additional variable u_5 using a fourth generator ψ for an extension to the 5-dimensional case). Another possible structure of a PND copula is illustrated in figure 1.

Thirdly, copula C from (2.3) is also an example of a so-called hierarchical Archimedean (HA) copulas. The basic idea of this approach (see, e.g. Savu & Trede, 2006) is to build a hierarchy of Archimedean copulas. Let there be L hierarchy levels indexed by l. At each level $l = 1, \dots, L$ one has n_l distinct objects with index $j = 1, \dots, n_l$. The $u_1, \dots, u_d$ are located at the lowest level, $l = 0$. At level $l = 1$ the $u_1, \dots, u_d$ are grouped into n_1 ordinary multivariate Archimedean copulas $C_{1,j}$, $j = 1, \dots, n_1$, of the form

$$C_{1,j}(\mathbf{u}_{1,j}) = \varphi_{1,j}^{-1}\left(\sum \varphi_{1,j}(\mathbf{u}_{1,j})\right)$$

where $\varphi_{1,j}$ denotes the generator of copula $C_{1,j}$. Let $\mathbf{u}_{1,j}$ denote the set of elements of $u_1, \dots, u_d$ belonging to copula $C_{1,j}$ for $j = 1, \dots, n_1$. The copulas $C_{1,1}, \dots, C_{1,n_1}$ might belong to different Archimedean families. All copulas of level $l = 1$ are in turn aggregated into copulas at level $l = 2$. The n_2 copulas $C_{2,j}$, $j = 1, \dots, n_2$ are generalized Archimedean copulas, whose dependence structure is only of partial exchangeability. They consist of copulas from the previous level (as elements) and can be represented as

$$C_{2,j}(\mathbf{C}_{2,j}) = \varphi_{2,j}^{-1}\left(\sum_{\mathbf{C}_{2,j}} \varphi_{2,j}(\mathbf{C}_{2,j})\right),$$

where $\varphi_{2,j}$ denotes the generator of copula $C_{2,j}$, and $\mathbf{C}_{2,j}$ represents the set of all copulas from level $l = 1$ entering copula $C_{2,j}$ for $j = 1, \dots, n_2$. We can proceed in this manner until attaining level L with the hierarchical Archimedean copula $C_{L,1}$ as single object.

In order to ensure that $C_{L,1}$ is a proper copula, we have to proclaim that $\varphi_{l,j}^{-1} \in \mathcal{L}_\infty$ for $l = 1, \dots, L$ and $j = 1, \dots, n_l$, and that $\varphi_{l+1,i} \circ \varphi_{l,j}^{-1} \in \mathcal{L}_\infty^*$ for all $l = 1, \dots, L$ and $j = 1, \dots, n_l$, $i = 1, \dots, n_{l+1}$ such that $C_{l,j} \in \mathbf{C}_{l+1,i}$. Moreover, a hierarchy is established if the number of copulas decreases at each level, if the top level contains only a single object and if at each level the dimensions of the copulas add up to d. Figure 2 displays the possible construction of a 5-dimensional HA-copula.

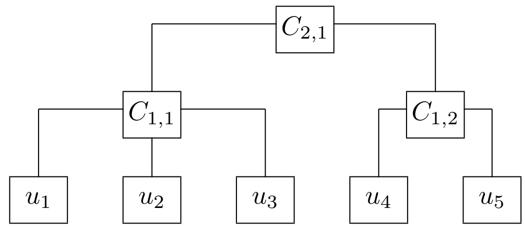

Figure 2: HA copula for $d = 5$.

Savu & Trede (2006) also derive the HA-copula density

$$\frac{\partial^d C_{L,1}}{\partial u_1 \dots \partial u_d} = \sum \frac{\partial^{d-i} C_{L,1}}{\partial C_{L-1,1}^{k_1} \cdots \partial C_{L-1,n_{L-1}}^{k_n,L-1}} \prod_{r=1}^{n_{L-1}} \sum_{u=\upsilon_1,\dots,\upsilon_r} \frac{\partial^{|\upsilon_1|} C_{L-1,r}}{\partial \upsilon_1}, \dots, \frac{\partial^{|\upsilon_r|} C_{L-1,r}}{\partial \upsilon_r},$$

where the outer sum extends over all sets of integers $k_1, \dots, k_{n,L-1} \in \mathbb{N}_0$ such that $\max_j k_j \leq d_{L-1,j}$ and $\sum_{j=1}^{n_{L-1}} k_j = d - i$ for all $i = 0, \dots, d - n_{L-1}$. These terms are the outer derivatives of the copula with respect to the elements of $\mathbf{C}_{L,1}$, i.e., the n_{L-1} copulas from level $L-1$. The second part of the formula are the inner derivatives, corresponding to the derivatives of the copulas at level $L-1$ with respect to their arguments $\mathbf{u}_{L-1,j}$.

2.1.3 Generalized multiplicative Archimedean copulas

In this section we focus on methods recently proposed by Morillas (2005) and Liebscher (2006). Both approaches are based on a second functional representation of Archimedean copulas via so called multiplicative generators (see Nelsen, 2006). Setting $\vartheta(t) \equiv \exp(-\varphi(t))$ and $\vartheta^{[-1]}(t) \equiv \varphi^{[-1]}(-\ln t)$, equation (2.1) can be rewritten as

$$C(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(\vartheta(u_1) \cdot \vartheta(u_2) \cdot \dots \cdot \vartheta(u_d)\Big). \tag{2.4}$$

The function ϑ is called multiplicative generator of C. Due to the relationship between φ and ϑ, the function $\vartheta : [0,1] \to [0,1]$ is continuous, strictly increasing and concave with $\vartheta(1) = 1$ and $\vartheta^{[-1]}(t) = 0$ if $0 \leq t \leq \vartheta(0)$ and $\vartheta^{[-1]}(t) = \vartheta^{-1}(t)$ if $\vartheta(0) \leq t \leq 1$.
Equation (2.4) can also be expressed using the independence copula $C^{\perp}(\mathbf{u}) = \prod_{i=1}^{d} u_i$:

$$C(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(C^{\perp}(\vartheta(u_1), \dots, \vartheta(u_d))\Big).$$

Morillas (2005) substitutes $C^{\perp}$ by an arbitrary d-copula C in order to obtain

$$C_{\vartheta}(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(C(\vartheta(u_1), \vartheta(u_2), \dots, \vartheta(u_d))\Big) \tag{2.5}$$

and proves that C_{ϑ} is a d-copula if $\vartheta^{[-1]}$ is *absolutely monotonic of order* d on $[0,1]$, i.e. if $\vartheta^{[-1]}(t)$ satisfies $(\vartheta^{[-1]})^{(k)}(t) = \frac{d^k \vartheta^{[-1]}(t)}{dt^k} \geq 0$ for $k = 1, 2, \dots, d$ and $t \in (0,1)$.

Examples of generator functions are stated in Morillas (2005). Notice that not every generator given there is absolutely monotonic for arbitrary $d > 1$: As one can easily verify, the generator $\vartheta(t) = t^r/(2-t^r), r \in (0, 1/3]$ (see table 1, no. 9 in Morillas, 2005) has no absolutely monotonic pseudo-inverse of order $d \geq 3$, because the third derivative of $\vartheta^{[-1]}$ becomes negative. Hence this generator is suitable only for a construction of generalized bivariate copulas. Concerning the basic properties of such Morillas copulas we refer to Morillas (2005).

Another way of generalizing Archimedean copulas is the method proposed by Liebscher (2006) who introduces the following copula representation

$$C(u_1, \ldots, u_d) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{j1}(u_1) \cdot \psi_{j2}(u_2) \cdot \ldots \cdot \psi_{jd}(u_d)\right), \tag{2.6}$$

where Ψ and $\psi_{jk} : [0, 1] \to [0, 1]$ are functions satisfying the following conditions: Firstly, it is assumed that $\Psi^{(d)}$ exist with $\Psi^{(k)}(u) \geq 0$ for $k = 1, 2, \ldots, d$ and $u \in [0, 1]$, and that $\Psi(0) = 0$. Secondly, ψ_{jk} is assumed to be differentiable and monotone increasing with $\psi_{jk}(0) = 0$ and $\psi_{jk}(1) = 1$ for all k, j. Thirdly, Liebscher's construction requires that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{jk}(v)\right) = v \quad \text{for} \;\; k = 1, 2, \ldots, d \text{ and } v \in [0, 1].$$

The three conditions guarantee that C defined in (2.6) is actual a copula.

It is easily seen that the approaches of Morillas (2005) and Liebscher (2006) coincide for $m = 1$, $\vartheta^{[-1]} = \Psi$ in (2.6) and $C_\vartheta = C^\perp$ in (2.5).

Liebscher (2006) also states a general method for deriving appropriate functions ψ_{jk}. Let $h_{jk} : [0, 1] \to [0, 1], j = 1, \ldots, m, k = 1, \ldots, d$ be a differentiable and bijective function such that $h'_{jk}(u) > 0$ for $u \in (0, 1)$, $h_{jk}(0) = 0$, $h_{jk}(1) = 1$ and $mu = \sum_{j=1}^{m} h_{jk}(u)$, $u \in [0, 1]$, $k = 1, \ldots, d$. Let $\psi = \Psi^{-1}$ be the differentiable inverse function of Ψ. An appropriate choice is setting $\psi_{jk}(u) = h_{jk}(\psi(u))$, since $\psi'_{jk}(u) = h'_{jk}(\psi(u)) \cdot \psi'(u) > 0$ for $j = 1, \ldots, m$ and $u \in [0, 1]$.
Considering $m = 2$, define

$$h_{1k}(u) \equiv u^{\delta_k}, \quad h_{2k}(u) \equiv 2u - u^{\delta_k} \quad \text{with} \;\; \delta_k \in [1, 2]. \tag{2.7}$$

Choosing further

$$\Psi(t) = -\frac{1}{\theta}\ln(1 - (1 - e^{-\theta})t), \;\; \text{and} \;\; \psi(u) = \frac{1 - e^{-\theta u}}{1 - e^{-\theta}}, \; \theta > 0$$

and defining $\psi_{jk}(u) \equiv h_{jk}(\psi(u))$ a generalized Frank copula (GMLF) is obtained. Setting $\delta_k = 1$ for all but one k, $k = 1, \ldots, d$, it is easily verified that the GMLF copula reduces to the common Frank copula. Setting $m = 2$ and h_{jk} as in (2.7) but now choosing (see table 2, no. 2, p. 8 in Liebscher (2006))

$$\Psi(t) = \frac{(\delta - t)^{-\theta} - \delta^{-\theta}}{(\delta - 1)^{-\theta} - \delta^{-\theta}}, \quad \text{and} \quad \psi(u) = \delta - \left(\delta^{-\theta}(1-u) + u(\delta-1)^{-\theta}\right)^{-\frac{1}{\theta}}, \ \theta > 0, \delta > 1$$

a copula is obtained, which will be termed as GML2 copula henceforth.

In the field of insurance pricing the function ψ_{jk} is known as a distortion function (for a definition see Freez & Valdez, 1998) and the methods proposed by e.g. Freez & Valdez (1998) or Wang (1998) appear as special cases in (2.6). The same holds for the approach given by Morillas (2005) where the function ϑ also satisfies the requirements of a distortion function.

2.2 Pair-copula decompositions

2.2.1 Pair-copula decomposition: The general case

One way of calculating a multivariate density is by decomposing it into a product of marginal densities and conditional densities. The latter can be stepwise replaced by so-called pair-copulas. Again, let $\mathbf{X} = (X_1, \ldots, X_d)'$ have the joint density function

$$f(x_1, \ldots, x_d) = f(x_d) \cdot f(x_{d-1}|x_d) \cdot f(x_{d-2}|x_{d-1}, x_d) \cdot \ldots \cdot f(x_1|x_2, \ldots, x_d) \tag{2.8}$$

which is unique up to a relabelling of the variables. Because of

$$f(x_1, \ldots, x_d) = c_{12\cdots d}(F_1(x_1), \ldots, F_d(x_d)) \cdot f_1(x_1) \cdots f_d(x_d),$$

with $c_{12\cdots d}(\cdot)$ being the d-variate copula density, $f(x_d|x_{d-1})$, e.g., may also be expressed by $c_{12}(F_1(x_1), F_2(x_2)) \cdot f_1(x_1)$. $c_{12}(\cdot, \cdot)$ is called *pair-copula* density for the respecting transformed variables. $f(x_{d-2}|x_{d-1}, x_d)$, again, can be decomposed into

$$c_{(d-2)d|d-1}\left(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1})\right) \cdot f(x_{d-2}|x_{d-1}).$$

Using $f(x_{d-2}|x_{d-1}) = c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2})$ results in

$$\begin{aligned} f(x_{d-2}|x_{d-1}, x_d) &= c_{(d-2)d|d-1}(F_{d-2|d-1}(x_{d-2}|x_{d-1}), F_{d|d-1}(x_d, x_{d-1})) \\ &\cdot \; c_{(d-2)(d-1)}(F_{d-2}(x_{d-2}), F_{d-1}(x_{d-1})) \cdot f_{d-2}(x_{d-2}). \end{aligned}$$

But this is not unique anymore, because, while splitting up, conditioning on x_d instead of x_{d-1} is also possible. This leads to a different decomposition. The *general formula* reads

$$f(x|\mathbf{v}) = c_{xv_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}, F(v_j|\mathbf{v}_{-j})) \cdot f(x|\mathbf{v}_{-j}) \tag{2.9}$$

for a d-dimensional vector v with components v_j. $\mathbf{v}_{-j}$ denotes v excluding the component v_j. For methods and formulas to calculate $F(x|\mathbf{v})$ we refer to Joe (1996).

As seen above, every (conditional) d-dimensional density can be split up into a pair-copula and a $(d-1)$-dimensional (conditional) density. For $d > 2$ you can iteratively repeat this splitting for the $(d-1)$-dimensional conditional density. Eventually, you will get a product of univariate densities and pair-copulas. Like shown in the trivariate case, this decomposition is not unique but there are various ways to do so.

In order to sort the different decomposition constructs, so-called *regular vines* (see Bedford and Cooke, 2001 and 2002) are defined. Vines are graphical models that present complete decomposition schemes. Following Aas et al. (2006) we choose the structure of the D-vine, since there is no dominating variable. The joint density $f(x_1, \ldots, x_d)$ can be expressed as

$$\prod_{k=1}^{d} f(x_k) \prod_{j=1}^{d-1} \prod_{i=1}^{d-j} c_{i,i+j|i+1,\ldots,i+j-1}(F(x_i|x_{i+1}, \ldots, x_{i+j-1}), F(x_{i+j}|x_{i+1}, \ldots, x_{i+j-1})).$$

The decomposition of a four-dimensional density according to the D-vine scheme is

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4) \\ &\cdot c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \cdot c_{34}(F(x_3), F(x_4)) \\ &\cdot c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3)) \\ &\cdot c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3)). \end{aligned} \tag{2.10}$$

2.2.2 Pair-copula decomposition of a copula

Originally, the pair-copula decomposition (PCD) decomposes the common density f of d random variables. Of course, one may also apply the pair-copula decomposition to the underlying copula density c, as we will show in this subsection. To simplify notation, we restrict ourselves to $d = 4$ and the D-vine decomposition. As an immediate consequence of Sklar's (1959) theorem,

$$c(F(x_1), F(x_2), F(x_3), F(x_4)) = \frac{f(x_1, x_2, x_3, x_4)}{f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot f(x_4)}.$$

Substituting the common density by its PCD given in (2.10),

$$\begin{aligned} c(F(x_1), F(x_2), F(x_3), F(x_4)) &= c_{12}(F(x_1), F(x_2)) \cdot c_{23}(F(x_2), F(x_3)) \cdot c_{34}(F(x_3), F(x_4)) \\ &\cdot c_{13|2}(F(x_1|x_2), F(x_3|x_2)) \cdot c_{24|3}(F(x_2|x_3), F(x_4|x_3)) \\ &\cdot c_{14|23}(F(x_1|x_2, x_3), F(x_4|x_2, x_3)) \end{aligned}$$

with $c_{i|j}(\cdot, \cdot)$ being a pair-copula density and its indices i, j refer to x_i and x_j. According to Joe (1996),

$$F(x|\mathbf{v}) = \frac{\partial\, C_{x,v_j|\mathbf{v}_{-j}}(F(x|\mathbf{v}_{-j}), F(v_j|\mathbf{v}_{-j}))}{\partial\, F(v_j|\mathbf{v}_{-j})}$$

with $\mathbf{v}_{-j}$ being the vector $\mathbf{v}$ except the element v_j. In the univariate case (i.e. $\mathbf{v} = v$),

$$F(x|v) = \frac{\partial\, C_{x|v}(F_X(x), F_V(v))}{\partial\, F_V(v)} \equiv h(x, v, \theta),$$

where θ being the parameter vector of the copula $C_{x,v}$. The copula density decomposition can be written as follows: It is obvious that $F(x_1|x_2) = h(x_1, x_2, \theta_{12})$ with θ_{12} is the parameter (vector) of the of copula C_{12}. Analogously, $F(x_3|x_2) = h(x_3, x_2, \theta_{23})$, $F(x_2|x_3) = h(x_2, x_3, \theta_{23})$ and $F(x_4|x_3) = h(x_4, x_3, \theta_{34})$. $F(x_1|x_2, x_3)$, again, can be iteratively simplified to

$$\frac{\partial\, C_{13|2}(F(x_1|x_2), F(x_3|x_2))}{\partial\, F(x_3|x_2)} = h(h(x_1, x_2, \theta_{12}), h(x_3, x_2, \theta_{32}), \theta_{13|2}).$$

Analogously, $F(x_4|x_2, x_3)$ can be written as

$$\frac{\partial\, C_{24|3}(F(x_4|x_3), F(x_2|x_3))}{\partial\, F(x_2|x_3)} = h(h(x_4, x_3, \theta_{43}), h(x_2, x_3, \theta_{23}), \theta_{24|3}).$$

Finally, define $u_1 = F(x_1)$, $u_2 = F(x_2)$, $u_3 = F(x_3)$, $u_4 = F(x_4)$. The formula for the 4-dimensional PCD copula density now reads as

$$\begin{aligned} c(\mathbf{u}) \;=\; & c_{12}(u_1, u_2) \cdot c_{23}(u_2, u_3) \cdot c_{34}(u_3, u_4) \\ & \cdot c_{13|2}(h(u_1, u_2, \theta_{12}), h(u_3, u_2, \theta_{23})) \cdot\; c_{24|3}(h(u_2, u_3, \theta_{23}), h(u_4, u_3, \theta_{34})) \\ & \cdot c_{14|23}(h(h(u_1, u_3, \theta_{13}), h(u_2, u_3, \theta_{23}), \theta_{13|2}), h(h(u_4, u_3, \theta_{43}), h(u_2, u_3, \theta_{23}), \theta_{24|3})) \end{aligned}$$

To summarize, in order to specify a d-dimensional (copula) density, two main steps have to be taken (see Aas et al., 2006): Firstly, an appropriate decomposition scheme has to be selected as follows: use the canonical vine scheme, if there is a key-variable. If not, use the D-vine scheme. Secondly, the pair-copulas have to be specified: e.g. Gaussian, Student's t, Archimedean or Gumbel copula. It is possible, to use one copula model for all pair-copulas or decide individually.

2.3 Koehler-Symanowski (KS) copulas

Koehler & Symanowski (1995) introduce a multivariate distribution as follows: With the index set $V = \{1, 2, \ldots, d\}$, $\mathcal{V}$ being the power set of V and $\mathcal{I} \equiv \{I \in \mathcal{V}$ with $|I| \geq 2\}$ let $\mathbf{X}$ denote a d-dimensional random vector with univariate marginal distributions $F_i(x_i), i \in V$. For all subsets $I \in \mathcal{I}$ let $\alpha_I \in \mathbb{R}_0^+$ and $\alpha_i \in \mathbb{R}_0^+$ for all $i \in V$ such that $\alpha_{i+} = \alpha_i + \sum_{I \in \mathcal{I}} \alpha_I > 0$ for $i \in I$. Then the common cdf F is defined by

$$F(x_1, \ldots, x_d) = \frac{\prod_{i \in V} F_i(x_i)}{\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} F_j(x_j)^{\alpha_{j+}} - (|I| - 1) \prod_{i \in I} F_i(x_i)^{\alpha_{i+}}\right]^{\alpha_I}} \,. \tag{2.11}$$

The terms $K_I = \sum_{i \in I} \prod_{j \in I, j \neq i} F_j(x_j)^{\alpha_{j+}} - (|I| - 1) \prod_{i \in I} F_i(x_i)^{\alpha_{i+}}$ are called association terms. Moreover, Koehler & Symanowski (1995) showed that the joint density function exists if the marginal density functions f_i exist for all $i \in V$. Due to the design of the Koehler-Symanowski distribution the corresponding copula has a similar functional form: Setting $u_i = F_i(x_i)$ for all $i \in V$, the KS copula is

$$C(u_1, \ldots, u_d) = \frac{\prod_{i \in V} u_i}{\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} u_j^{\alpha_{j+}} - (|I| - 1) \prod_{i \in I} u_i^{\alpha_{i+}} \right]^{\alpha_I}} . \tag{2.12}$$

In contrast to the cumulative distribution function the functional representation of the density is quite complicated due to complex factors with additive components. Koehler & Symanowski (1995) gave an explicit formula for the special case of a so called KS(2)-distribution (Caputo, 1998), where all parameters α_I are set equal zero for $|I| > 2$. The corresponding copula will be termed as KS(2) copula henceforth. Assuming that $\alpha_{ij} \equiv \alpha_{ji} \geq 0$ for all $(i, j) \in V \times V$ and $\alpha_{i+} = \alpha_{i1} + \alpha_{i2} + \cdots + \alpha_{id} > 0$ for all $i \in V$, the KS(2)-copula simplifies to

$$C(u_1, u_2, \ldots, u_d) = \prod_{i=1}^{d} u_i \prod\prod_{i<j} K_{ij}^{-\alpha_{ij}} \tag{2.13}$$

with $K_{ij} \equiv u_i^{1/\alpha_{i+}} + u_j^{1/\alpha_{j+}} - u_i^{1/\alpha_{i+}} u_j^{1/\alpha_{j+}} = K_{ji}$.

Palmitesta & Provasi (2005) apply this particular KS copula to weekly log-returns. They also argue that this copula has the ability to model complex dependence structures among subsets of marginal distribution but they do not present any goodness-of-fit measure or any comparison with other copulas. We seize the proposal by Palmitesta & Provasi but set the association parameter $\alpha_I \geq 0$ for $|I| = 2$ and $|I| = 4$ in (2.12), while all parameters α_I are set equal to zero for $|I| = 3$, i.e. we include a global dependence parameter and refer to this copula as augmented KS(2) copula (aKS(2)). Additionally, we use the KS-Copula (KSC) as defined in (2.12).

2.4 Multiplicative Liebscher copulas

By now, different methods have been reviewed how to construct d-variate copulas. Liebscher (2006), in contrast, discusses how to combine or connect a given set of k possibly different d-copulas $C_1, \ldots, C_k$ to a new d-copula C in order to increase flexibility and/or introduce asymmetry. His proposal focusses on multiplicative connections of d-copulas of the form

$$C(u_1, \ldots, u_d) = \prod_{j=1}^{k} C_j(g_{j1}(u_1), \ldots, g_{jd}(u_d)) \tag{2.14}$$

with a set of $k \cdot d$ admissible functions $g_{11}, \ldots, g_{1d}, \ldots, g_{k1}, \ldots, g_{kd}$, each of which being bijective, monotonously increasing or identically equal 1 satisfying

$$\prod_{j=1}^{k} g_{ji}(v) = v, \quad i = 1, \ldots, d. \tag{2.15}$$

Note that (2.15) reduces to $g_{1i}(v) = v$ for $k = 1$ and $i = 1, \ldots, d$, and C is recovered. In accordance to Liebscher (2006), possible choices are

$$g_{ji}(v) \equiv v^{\theta_{ji}} \text{ with } \theta_{ji} > 0 \text{ and } \sum_{j=1}^{k} \theta_{ji} = 1 \text{ for } i = 1, \ldots, d \tag{2.16}$$

$$g_{1i}(v) = f(v), \quad g_{2i}(v) \equiv v \cdot \frac{1}{f(v)}, \quad f(v) = \left(\frac{1 - e^{-\theta_i v}}{1 - e^{-\theta_i}} \right)^{\alpha}, \quad \theta > 0,\ \alpha \in (0,1) \tag{2.17}$$

We consider four different generalized Clayton copulas based on (2.14). The "Generalized Clayton of Liebscher type I" (L_1) is obtained by setting $k = 2$, choosing the clayton copula for C_1, the independence copula for C_2 and $g_{ji}(v)$ as in (2.16). Applying (2.17) rather than (2.16), the "Generalized Clayton of Liebscher type II" (L_2) copula with $d + 2$ dependence parameters is constructed. Similarly, combining two d-variate Clayton copulas and using g from (2.16) we obtain the d-dimensional copula family with $d + 2$ parameters, termed as the "Generalized Clayton of Liebscher type III" (L_3) in the sequel. Finally, applying again (2.17) rather than (2.16), the "Generalized Clayton of Liebscher type IV" (L_4) is obtained.

3 Goodness-of-fit measures

We now tackle the problem to compare the goodness-of-fit (GOF) of the different copula models from section 2, noting that most of them are not nested. As we apply maximum likelihood (ML) methods to obtain estimators for the unknown parameter vector, the first choice is the log-likelihood value ℓ or – in order to take the different numbers of parameters in account – the information criterion of Akaike $AIC = -2\ell + (2N(K+1))/(N-K-2)$, where K and N denote the number of parameters to be fitted and the number of observations, respectively. However, comparing log-likelihood values for non-nested models may produce misleading conclusions. Therefore, certain GOF tests may come to application. Following Breymann, Dias & Embrechts (2003), Chen, Fan & Patton (2004) or recently Berg & Bakken (2006), the main idea is to project the multivariate problem into a set of independent and uniform $U(0,1)$ variables, given the multivariate distribution and to calculate the distance (e.g. Anderson-Darling, Kolmogorov-Smirnov, Cramér-von Mises, Kernel smoothing) between the transformed variables and the uniform distribution. In contrast to the authors above, we are not primarily interested whether the data stem from the specified copula model but we use these distances as citerion itself. The proceeding is roughly as follows:

By means of the Rosenblatt (1952) transformation the random vector $\boldsymbol{X} = (X_1, \ldots, X_d)'$ is mapped onto a random vector $\boldsymbol{Z}^* = (Z_1^*, \ldots, Z_d^*)'$ via

$$Z_1^* \equiv F_1(X_1) \text{ and } Z_i^* \equiv F_{X_i|X_1,\ldots,X_{i-1}}(X_i|X_1, \ldots, X_{i-1}), \quad i = 2, \ldots, d. \tag{3.1}$$

It can be shown that $\boldsymbol{Z}^*$ is uniformly distributed on $[0,1]^d$ with independent components $Z_1^*, \ldots, Z_d^*$. Assume that the cumulative distribution function of $\boldsymbol{X}$ admits the decomposition

$$F_{\boldsymbol{X}}(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)),$$

where $C(\cdot)$ denotes a parametric copula which is the common distribution function of $\boldsymbol{U} = (U_1, \ldots, U_d)'$ with $U_i \equiv F_{X_i}(X_i)$. Define $C(u_1, \ldots, u_j) \equiv C(u_1, \ldots, u_j, 1, \ldots, 1)$ for $j \leq d$. Furthermore, the conditional distribution of $U_i|U_1, \ldots, U_{i-1}$ is given by

$$C_i(u_i) \equiv \frac{\partial^{i-1} C(u_1, \ldots, u_i)}{\partial u_1 \ldots \partial u_{i-1}} \Big/ \frac{\partial^{i-1} C(u_1, \ldots, u_{i-1})}{\partial u_1 \ldots \partial u_{i-1}}.$$

According to (3.1), the variables

$$Z_1 \equiv C(U_1) = U_1 \quad \text{and} \quad Z_i \equiv C_i(U_i), \quad i = 2, \ldots, d \tag{3.2}$$

are independent and uniform on $[0,1]$. Consequently, the sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_N$ from a parametric copula and with marginals given by $F_1, \ldots, F_d$ can be mapped onto an *iid* sample $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_N$ from a uniform distribution on $[0,1]^d$.

Breymann et al. (2003) suggest to transform each random vector $\boldsymbol{Z}_i = (Z_{i1}, \ldots, Z_{id})'$ in a (univariate) chi-square variable χ_j with d degrees of freedom through $\chi_j = \sum_{i=1}^d \Phi^{-1}(Z_{ji})^2$, $j = 1, \ldots, N$, where $\Phi^{-1}(u)$ denotes the standard normal quantile function. If the margins are unknown, they may be replaced by the corresponding empirical counterparts. Breymann et al. state that "we do assume that the χ^2-distribution will not be significantly affected by the use of the empirical distribution functions used to transform the marginal data".

4 The data set

The data sets we used used to compare the different copula models come from three different markets (German stock market, foreign exchange (FX) market and commodity markets). From each market, four typical representatives were selected, provided that the corresponding sample period is sufficiently large. Instead of analyzing the prices themselves, we calculated and considered (percentual) continuously compounded returns ("log-returns") $R_t = 100(\log P_t - \log P_{t-1}), \quad t = 2, \ldots, N$. In order to account for possible time-dependencies (which are common to most financial return series), we also fitted univariate GARCH models of the form $R_t = \mu + \gamma_1 R_{t-1} + \ldots + \gamma_k R_{t-k} + h_t \epsilon_t$ with variance equations $h_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \ldots + \alpha_1 R_{t-p}^2 + \beta_1 h_{t-1}^2 + \ldots + \beta_q h_{t-q}^2$ to each of the series and considered standardized residuals ϵ_t rather the original returns R_t. Secondly, as we are primarily not interested in parametric models for the marginal distributions, all observations (i.e. returns or standardized residuals) were transformed into uniform ones by means of the (empirical) probability integral transform, i.e.

$$U_t = F_N(R_t) \text{ with } F_N(x) = \frac{\{\#R_t | R_t \leq x\}}{\#R_t} \text{ and } U_t^* = F_N(\epsilon_t).$$

4.1 German stock returns

From the German stock market, we selected prices of HVB AG, BMW AG, Allianz AG and Munich Re AG, all of them being part of the German stock market index DAX which measures the performance of the Prime Standard's 30 largest German companies in terms of order book volume and market capitalization. Figure 3 contains the series of prices and returns. Table 1 summarizes descriptive and inductive statistics. All series feature negative

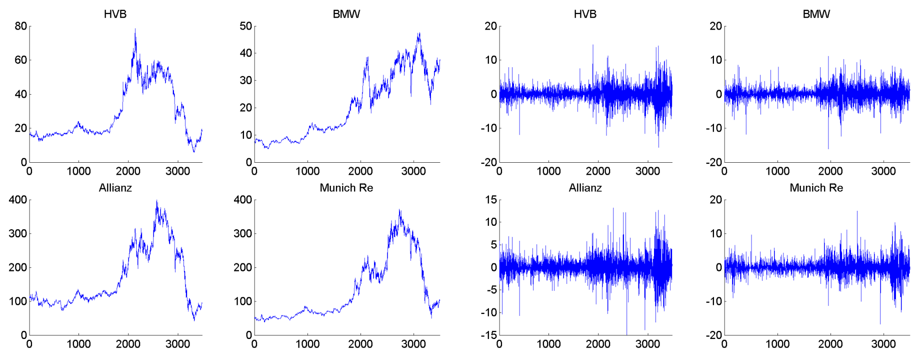

Figure 3: German stock prices and stock returns.

skewness and high kurtosis (measured by the third and fourth standardized moment $\mathbb{S}$ and $\mathbb{K}$). Morerover, there is empirical evidence for (slight) serial correlation and GARCH effects as the Ljung-Box statistic $\mathcal{LB}$ and Engle's Lagrange multiplier statistic $\mathcal{LM}$ indicate.

Start	End	N	Stocks	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
02-01-90	12-11-03	3486	HVB	0.004	5.61	-0.033	8.16	24.45	621.08
02-01-90	12-11-03	3486	BMW	0.046	4.33	-0.132	7.19	28.96	366.49
02-01-90	12-11-03	3486	Allianz	-0.002	4.87	-0.07	8.37	29.74	517.14
02-01-90	12-11-03	3486	MunichRe	0.02	5.06	-0.027	8.75	50.53	508.59

Table 1: German stock returns.

4.2 Exchange rate returns

Data from foreign exchange markets (FX-markets) are available from the PACIFIC Exchange Rate Service[1]. This service offered by Prof. Werner Antweiler at UBC's Sauder School of Business provides access to current and historic daily exchange rates through an on-line database retrieval and plotting system. In contrast to the volume notation, where values are expressed in units of the target currency per unit of the base currency, the price

[1]Download under the URL-link `http://pacific.commerce.ubc.ca`.

notation is used within this work which corresponds to the numerical inverse of the volume notation. All values are expressed in units of the base currency (here US-Dollar) per unit of the target currency. Table 2 summarizes the statistics of the four exchanges rates (Canadian Dollar, Japanese Yen, Swiss Franc, British Pound) which are used later on. Again, prices and log-returns in figure 4, below.

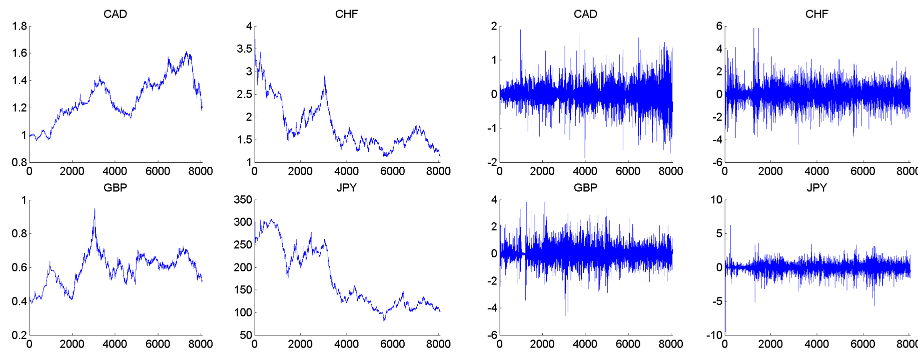

Figure 4: Exchange rates: Prices versus Returns.

Start	End	N	FX Rate	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
02-01-90	31-12-04	3794	CAD	0.002	0.09	-0.004	6.75	12.65	912.18
02-01-90	31-12-04	3794	YEN	-0.015	0.56	-0.002	6.11	12.5	429.24
02-01-90	31-12-04	3794	SFR	0.003	0.36	0.132	6.84	55.79	485.26
02-01-90	31-12-04	3794	BRP	-0.013	0.44	-0.723	13.33	34.48	176.20

Table 2: Exchange rates

4.3 Metal returns

The London Metal Exchange[2] (LME) is the world's premier non-ferrous metals market with a turnover value of some US$2000 billion per annum. For a detailed introduction on metal markets with emphasis on the London metal exchange see Crowson & Sampson (2001). Among the different metals, emphasis is placed on aluminium, copper, lead and nickel. All prices are quoted in US-Dollar per tonne. Table 3 contains again the basic summary statistics. Prices and log-returns are displayed in figure 5.

[2]Download under `http://www.lme.co.uk/`.

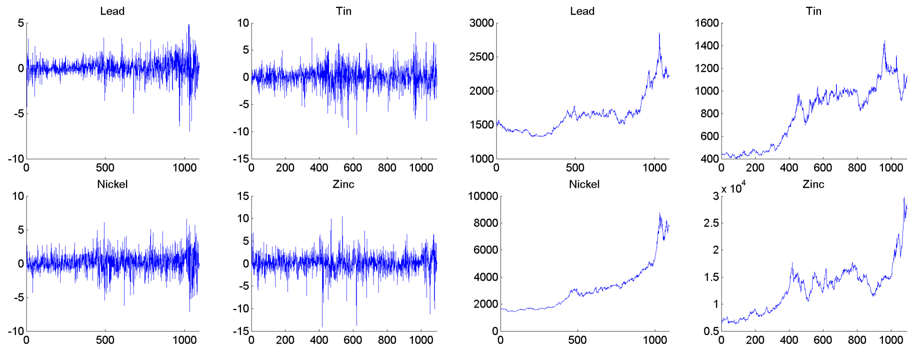

Figure 5: Metals: Prices versus Returns.

Start	End	N	Metal	μ	s^2	$\mathbb{S}$	$\mathbb{K}$	$\mathcal{LB}(10)$	$\mathcal{LM}(10)$
26-03-99	07-08-06	1093	Lead	0.034	1.22	-0.555	8.72	29.74	161.56
26-03-99	07-08-06	1093	Tin	0.084	4.21	-0.368	5.59	30.11	100.37
26-03-99	07-08-06	1093	Nickel	0.142	2.38	-0.139	5.13	12.95	127.66
26-03-99	07-08-06	1093	Zinc	0.13	4.97	-0.618	7.9	10.72	21.45

Table 3: Metals: Prices versus Returns.

5 Empirical results

The 4-copulas under consideration are the following: Firstly, we selected the Clayton copula (CLA), the Gumbel copula (GUM) and its rotated version (roGUM) from the Archimedean class. From the generalized Archimedean copula family, two hierarchical copula models (i.e. HA-CLA and HA-GUM) are included, based on the Clayton and the Gumbel copula, respectively. Moreover, six representatives of Morrillas' construction scheme (i.e. MO-CLA1, MO-CLA2, MO-CLA3, MO-GUM1, MO-GUM2, MO-GUM3) involving the Clayton, the Gumbel and different generator functions (no. 3, 2, 4 in Morillas, 2005) are included as well. In addition, two version of Liebscher's proposal (GMLF, GML2) are used. Above that, representing the "elliptical copula world", the Gaussian copula (NORM) and – as ultimate benchmark – the Student-t (T) copula are also included. From the pair-copula decomposition we chose five representatives (i.e. PC-NORM, PC-T, PC-CLA, PC-GUM, PC-roGUM) each of them derived from one single copula model (i.e. we used no decompositions based on different copulas). Finally, the KS(2)-copula and its augmented version (which is a generalized version of Palmitesta & Provasi, 2005 because we included a general dependence parameter) and four different types of multiplicative Liebscher copulas from example 2.8 (L_1, L_2, L_3, L_4) are considered.

The computer code for the ML-estimation was implemented in Matlab 7.1. For maximization purposes we used the line-search algorithm of Matlab. The following tables include the comprehensive results for the parameter estimates as well as the different goodness-of-fit measures for all data sets (returns and GARCH-residuals) and all copulas models mentioned above. As stated above, goodness-of-fit is measured by the Log-likelihood value and Akaike's AIC criterion. Above that, two distance measures,

$$\text{KS} = \sqrt{N} \max_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right| \quad \text{and}$$

$$\text{AKS} = \frac{1}{\sqrt{N}} \sum_{j=1,\ldots,N} \left| F_{\chi^2(d)}(\chi_j) - F_{N,\chi}(\chi_j) \right|$$

are calculated to quantify the distance after application of the Rosenblatt transformation (based on the different parametric copula models).

The estimation results are unique across the different data set. As known from several empirical studies, the fit of the 4-variate Gaussian distributions may be considerably improved if the 4-variate Student-t distribution is considered, instead. However, pair-copula decompositions based on bivariate Student-t copulas produce a similar goodness-of-fit, sometimes even outperform the 4-variate Student-t distribution. Whereas t-PCD dominate the likelihood criteria, 4-variate Student-t distribution provide minimal distance measure in many cases. Similar, PCD-decompositions based on Archimedean copulas may also be considered as possible alternatives. This also applies to the three copulas $L1$, $L3$ and $L4$, in particular for the exchange rate data, whereas all copulas based on Morillas' approach and, of course, the plain Archimedean copulas feature low goodness-of-fit measures. Considering hierarchical Archimedean copulas, instead, we found only slight improvement, at least for our data sets. However, we have to confess that one might improve the results with another hierarchy which might be found on the basis of cluster algorithms.

The KS(2)-copula (recommended by Palmitesta and Provasi, 2005) provides only a poor fit to the return series. However, introducing an additional dependence parameter – which quantifies the overall dependence in the data set – clearly improves all goodness-fit measures. Above that, removing the GARCH effects found in the margins doesn't change the estimation results substantially. In particular, the ordering of the goodness-of-fit measures is essentially preserved. Above that, parameter estimates of the dependence parameters are roughly stable for most of the copulas under consideration.

To sum up, the 4-variate Student-t distribution still plays a predominant role. Some of the recently proposed construction schemes are partially competitive while others are more likely to be overestimated in the literature.

	German stocks (GARCH-Residuals)				**Exchange rates** (GARCH-Residuals)				**Metal** (GARCH-Residuals)			
Copula	l	AIC	KS	AKS	l	AIC	KS	AKS	l	AIC	KS	AKS
CLA	1847.1	-3690.2	5.90	0.89	1780.5	-3556.9	5.53	0.89	425.9	-847.8	1.64	0.26
GUM	1835.0	-3666.0	4.87	0.85	1787.1	-3570.2	4.64	0.94	385.4	-766.8	1.15	0.25
roGUM	1950.3	-3896.7	4.98	0.80	1818.7	-3633.4	4.74	0.93	425.8	-847.6	1.14	0.24
NORM	2284.1	-4554.1	5.07	0.73	3565.0	-7115.9	4.45	0.72	546.1	-1078.1	1.26	0.18
T	2718.2	-5420.3	**1.21**	**0.20**	3873.5	-7731.0	**1.65**	0.16	567.0	-1117.8	**1.14**	**0.08**
PC-NORM	2284.0	-4553.9	5.09	0.73	3564.8	-7115.6	4.45	0.71	545.6	-1077.0	1.26	0.18
PC-T	**2748.6**	**-5471.0**	1.29	0.21	**3959.2**	**-7892.3**	1.67	**0.15**	**574.9**	**-1123.4**	**1.14**	**0.08**
PC-CLA	2014.6	-4015.1	5.51	0.84	2950.5	-5886.9	5.19	0.82	476.6	-939.1	1.45	0.23
PC-GUM	2281.4	-4548.7	3.94	0.52	3408.5	-6802.9	3.58	0.54	463.1	-912.1	1.15	0.16
PC-roGUM	2448.3	-4882.6	3.55	0.49	3561.4	-7108.9	3.14	0.47	544.9	-1075.7	1.12	0.10
GMLF	1922.0	-3832.1	4.49	0.75	2712.5	-5412.9	5.54	0.83	477.9	-943.8	1.25	0.19
GML2	1927.0	-3840.0	4.85	0.76	2712.4	-5410.9	5.54	0.83	477.9	-941.8	1.25	0.19
KS(2)	869.6	-1717.1	6.94	1.00	164.8	-307.6	6.81	1.13	143.2	-264.2	2.33	0.37
aKS(2)	2218.8	-4413.4	2.57	0.44	3440.7	-6857.4	2.79	0.41	493.9	-963.5	1.35	0.18
MO-CLA1	1899.1	-3792.1	5.72	0.81	1968.0	-3929.9	4.95	0.79	454.2	-902.3	1.47	0.20
MO-GUM1	2028.8	-4051.5	4.03	0.70	1992.4	-3978.7	4.39	0.78	454.1	-902.1	1.08	0.16
MO-CLA2	1847.1	-3688.2	5.90	0.89	1780.5	-3554.9	5.53	0.89	425.9	-845.8	1.64	0.26
MO-GUM2	1835.0	-3664.0	4.87	0.85	1787.1	-3568.2	4.64	0.94	385.4	-764.7	1.15	0.25
MO-CLA3	1899.1	-3792.1	5.72	0.81	1968.0	-3929.9	4.95	0.79	454.2	-902.3	1.47	0.21
MO-GUM3	2028.8	-4051.5	4.03	0.70	1992.4	-3978.7	4.39	0.78	454.1	-902.1	1.08	0.16
L1	1977.9	-3943.8	4.53	0.62	3044.9	-6077.7	4.52	0.56	463.6	-915.0	1.21	0.18
L2	1952.4	-3890.7	4.34	0.57	2400.0	-4785.9	5.16	0.76	456.6	-899.1	1.18	0.16
L3	2137.9	-4261.7	3.29	0.47	3105.6	-6197.1	3.39	0.47	523.4	-1032.7	0.99	0.15
L4	2209.8	-4403.6	3.06	0.46	2278.4	-4540.8	3.00	0.51	572.6	-1129.0	1.31	0.85
HA-CLA	1961.7	-3915.4	5.41	0.82	1805.9	-3603.9	5.32	0.88	430.4	-852.7	1.49	0.26
HA-GUM	2069.7	-4131.4	4.15	0.66	1914.8	-3821.7	4.40	0.84	402.3	-796.6	1.15	0.24

Table 4: Goodness-of-fit measures: German stock returns (left), Exchange rates (middle) and Metal returns (right)

Copula	θ	**Copula**	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	**Copula**	θ_{11}	θ_{12}	θ_{21}
CLA	0.6144 (0.014)	**PC-CLA**	0.5528 (0.0259)	0.6914 (0.0283)	0.9043 (0.0317)	0.5513 (0.0299)	0.1463 (0.0196)	0.0736 (0.0169)	**HA-CLA**	0.5995 (0.0258)	0.8822 (0.0313)	0.5784 (0.0156)
GUM	1.3834 (0.0099)	**PC-GUM**	1.3183 (0.0163)	1.4179 (0.0188)	1.569 (0.022)	1.3528 (0.0189)	1.0663 (0.0113)	1.0479 (0.0103)	**HA-GUM**	1.3843 (0.0152)	1.5807 (0.0196)	1.3591 (0.0108)
roGUM	1.386 (0.0099)	**PC-roGUM**	1.3456 (0.0163)	1.4463 (0.0186)	1.5884 (0.0204)	1.3737 (0.0185)	1.0749 (0.0108)	1.0344 (0.0097)				

Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
NORM	0.4227 (0.0128)	0.5552 (0.0101)	0.3987 (0.0131)	0.4892 (0.0119)	0.3729 (0.0138)	0.5757 (0.0098)						
T	0.4336 (0.0136)	0.5715 (0.0109)	0.4100 (0.0139)	0.5059 (0.0125)	0.3835 (0.0142)	0.5777 (0.0107)	9.552 (0.7541)					
PC-NORM	0.4226 (0.0125)	0.4891 (0.011)	0.5757 (0.01)	0.4408 (0.0126)	0.1281 (0.0169)	0.0921 (0.0173)						
PC-T	0.4339 (0.0137)	0.5058 (0.0127)	0.5808 (0.011)	0.4559 (0.0137)	0.131 (0.0171)	0.0956 (0.0172)	8.0352 (0.9684)	6.7061 (0.794)	8.4613 (1.2098)	8.0099 (0.9807)	24.5287 (6.9544)	35.7712 (6.2027)

Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{1234}
KS(2)	0.0236 (6.9075)	0.0755 (3.2428)	0.2118 (20.2879)	0.0607 (10.24)	0.1889 (13.1611)	0 (0.9989)	0 (0.9983)	0.1707 (6.7221)	0 (8.2005)	0.6044 (159.5285)	
aKS(2)	0.0724 (0.014)	0.0456 (0.01)	0.1002 (0.0102)	0.0208 (0.0086)	0.1431 (0.0193)	0.0553 (0.0095)	0.0157 (0.0101)	0.0022 (0.0025)	0.1105 (0.0104)	0.1084 (0.0156)	0.2257 (0.021)

Copula	θ	r	**Copula**	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
MO-CLA1	0.2018 (0.0356)	−4.3616 (0.5431)	**L1**	1.5195 (0.1053)	0.7407 (0.0235)	0.6392 (0.0218)	0.8774 (0.0234)	0.7129 (0.0256)		
MO-GUM1	1.1036 (0.012)	−3.6082 (0.338)	**L2**	1.9807 (0.2387)	0.3782 (0.0897)	0.8093 (0.5656)	0.229 (0.6215)	1 (0.914)	0.4788 (0.589)	
MO-CLA2	5.2074 (0.2035)	8.4759 (0.4707)	**L3**	7.488 (1.3976)	0.4813 (0.024)	0.3632 (0.041)	0.2951 (0.0568)	0.3609 (0.0452)	0.336 (0.0631)	
MO-GUM2	1.3834 (0.0106)	1.9918 (0.2346)	**L4**	0.4267 (0.1021)	9.2654 (9.6158)	0.8587 (0.0695)	0.9222 (1.3772)	1 (1.7041)	0.6831 (1.0343)	0.6486 (0.786)
MO-CLA3	0.2018 (0.0357)	0.8135 (0.0189)								
MO-GUM3	1.1036 (0.0112)	0.783 (0.0171)								

Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2
GMLF	1.6401 (0.0679)	1.4309 (0.2393)	2 (1.1658)	1.7019 (0.1502)	2.3486 (0.2313)	
GML2	1.6403 (0.4725)	2 (0.8798)	1.4311 (0.2861)	1.7021 (0.0117)	1.1057 (0.0005)	0 (0.0001)

Table 5: German stock returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)

Copula	θ	Copula	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Copula	θ_{11}	θ_{12}	θ_{21}
CLA	0.3935 (0.0082)	**PC-CLA**	0.1689 (0.015)	1.1176 (0.0237)	0.5314 (0.0178)	0.0609 (0.0115)	0.3904 (0.0177)	0.0233 (0.0105)	**HA-CLA**	0.3771 (0.6515)	0.5045 (0.0374)	0.377 (0.126)
GUM	1.2397 (0.0056)	**PC-GUM**	1.1115 (0.0084)	1.7145 (0.0157)	1.3244 (0.0112)	1.021 (0.0062)	1.278 (0.0113)	1.0052 (0.0049)	**HA-GUM**	1.2256 (0.8538)	1.3517 (0.0714)	1.2255 (0.125)
roGUM	1.2388 (0.0056)	**PC-roGUM**	1.1081 (0.0082)	1.7405 (0.0158)	1.3454 (0.0111)	1.0307 (0.0069)	1.266 (0.0111)	1.0123 (0.0048)				

Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
NORM	0.1639 (0.0105)	0.1551 (0.0109)	0.108 (0.0111)	0.6339 (0.0057)	0.5273 (0.0067)	0.4082 (0.0084)						
T	0.1743 (0.0116)	0.1669 (0.0115)	0.1184 (0.0117)	0.65 (0.0061)	0.5352 (0.0081)	0.4239 (0.0092)	8.703 (0.4331)					
PC-NORM	0.1639 (0.0107)	0.6337 (0.0056)	0.408 (0.0084)	0.0673 (0.0112)	0.3805 (0.0092)	0.0183 (0.0111)						
PC-T	0.1769 (0.0105)	0.6498 (0.0064)	0.4232 (0.0089)	0.069 (0.0108)	0.3755 (0.0094)	0.0198 (0.0112)	9.3036 (1.0517)	4.8148 (0.2966)	6.5581 (0.4622)	87.8777 (4.3947)	9.0407 (1.136)	24.1813 (2.7923)

Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{1234}
KS(2)	0.2111 (1305.265)	0.0661 (7.2804)	0.0379 (36.6331)	0 (1469.3175)	0.3087 (150.3804)	0 (2.3777)	0 (62.1385)	0.2392 (143.6814)	0 (16.9564)	9.9596 (3314.7046)	
aKS(2)	1.9981 (0.3743)	0.0202 (0.945)	0.064 (0.1006)	0.011 (0.1077)	0 (0.9794)	0.1354 (0.0458)	0.0608 (0.0066)	0.0052 (0.0988)	0.0218 (0.0034)	0.0784 (0.0862)	0.1782 (0.0882)

Copula	θ	r	Copula	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
MO-CLA1	0.17 (0.0152)	−1.3564 (0.1054)	**L1**	2.4135 (0.1343)	0.0996 (0.0093)	0.8245 (0.0155)	0.7739 (0.0146)	0.5515 (0.0147)		
MO-GUM1	1.0684 (0.0071)	−1.4938 (0.1086)	**L2**	8.5815 (10.8401)	0.8341 (0.3222)	0 (0.9844)	1 (1.3353)	1 (1.2314)	1 (1.3307)	
MO-CLA2	0.9156 (0.9182)	2.327 (2.3052)	**L3**	5.3145 (0.4291)	0.3182 (0.0223)	0.0318 (0.0057)	0.5539 (0.0215)	0.5277 (0.0203)	0.4895 (0.0156)	
MO-GUM2	1.2397 (0.0052)	2.0165 (0)	**L4**	8.39 (12.6215)	0.2163 (0.1711)	0.3331 (0.2716)	1 (1.018)	0 (1.1084)	0 (1.0861)	0 (0.9463)
MO-CLA3	0.17 (0.0152)	0.5756 (0.0197)								
MO-GUM3	1.0684 (0.007)	0.599 (0.0181)								

Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2
GMLF	1 (0.9136)	2 (1.212)	1.9964 (0.0337)	1.7433 (0.2099)	1.3952 (0.8066)	
GML2	1 (1.0057)	2 (1.0471)	1.9964 (0.0989)	1.7434 (0.3373)	1.3296 (0.0008)	0 (0.0066)

Table 6: Exchange rate returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)

Copula	θ	Copula	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	Copula	θ_{11}	θ_{12}	θ_{21}
CLA	0.5754 (0.0245)	**PC-CLA**	0.3962 (0.0434)	0.8561 (0.0553)	0.7863 (0.0537)	0.472 (0.05)	0.2067 (0.0357)	0.0672 (0.0321)	**HA-CLA**	0.7007 (0.0519)	0.5714 (0.0471)	0.558 (0.0266)
GUM	1.3322 (0.0165)	**PC-GUM**	1.2100 (0.0254)	1.5084 (0.0339)	1.4363 (0.033)	1.2669 (0.0288)	1.1263 (0.023)	1.0251 (0.0173)	**HA-GUM**	1.4248 (0.0296)	1.3727 (0.0278)	1.3127 (0.0183)
roGUM	1.3384 (0.0168)	**PC-roGUM**	1.2369 (0.027)	1.5525 (0.0365)	1.4864 (0.0346)	1.295 (0.0312)	1.1145 (0.023)	1.0386 (0.0186)				

Copula	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{23}	ρ_{24}	ρ_{34}	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6
NORM	0.322 (0.0248)	0.4762 (0.0209)	0.3063 (0.0252)	0.5571 (0.0179)	0.4336 (0.0217)	0.5124 (0.0189)						
T	0.3288 (0.0269)	0.4817 (0.0219)	0.3157 (0.0263)	0.5669 (0.0195)	0.4386 (0.0223)	0.5206 (0.02)	13.827 (2.6336)					
PC-NORM	0.3215 (0.025)	0.5564 (0.0196)	0.5119 (0.0207)	0.3784 (0.0261)	0.2076 (0.0297)	0.0684 (0.0299)						
PC-T	0.3317 (0.0056)	0.5676 (0.0164)	0.5236 (0)	0.3795 (0.0233)	0.2089 (0.03)	0.0725 (0.0278)	16.7797 (0.1879)	8.8769 (1.0373)	11.0683 (0.7293)	9.3863 (1.6013)	23.917 (1.4834)	17.4431 (0.1221)

Copula	a_{11}	a_{12}	a_{13}	a_{14}	a_{22}	a_{23}	a_{24}	a_{33}	a_{34}	a_{44}	a_{1234}
KS(2)	0.0908 (3.8398)	0.0531 (6.8008)	0.2886 (24.7079)	0.0677 (4.531)	0.2623 (29.164)	0 (1)	0 (1)	0.2478 (6.6496)	0 (1)	0.4461 (56.5439)	
aKS(2)	0.219 (2.1377)	0.0053 (0.1216)	0.066 (0.5615)	0.0011 (0.4344)	0.0971 (0.0977)	0.1251 (0.5866)	0.057 (0.289)	0 (0.9378)	0.0817 (0.3054)	0.1346 (0.667)	0.2334 (3.4168)

Copula	θ	r	Copula	γ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
MO-CLA1	0.2688 (0.0597)	−2.7118 (0.6619)	**L1**	1.3437 (0.1788)	0.5905 (0.0464)	0.8287 (0.0444)	0.8988 (0.0385)	0.6683 (0.0481)		
MO-GUM1	1.0644 (0.0177)	−3.7782 (0.6788)	**L2**	1.7533 (0.3966)	0.3829 (0.0932)	0 (0.7151)	1 (1.3252)	1 (1.0031)	0.6321 (0.6753)	
MO-CLA2	11.1088 (0.5165)	19.3069 (1.3375)	**L3**	159.7481 (10.5399)	0.5683 (0.0316)	0.1413 (0.0094)	0.1205 (0.0082)	0.114 (0.0078)	0.1144 (0.0076)	
MO-GUM2	1.3322 (0.0166)	2.0028 (1)	**L4**	321.7142 (30.8244)	0.6025 (0.0445)	0.1503 (0.05)	0.4668 (1.1588)	0.9303 (1.1752)	0.9849 (1.1765)	1 (1.1737)
MO-CLA3	0.2688 (0.0591)	0.7306 (0.0476)								
MO-GUM3	1.0644 (0.0177)	0.7908 (0.0302)								

Copula	δ_1	δ_2	δ_3	δ_4	θ_1	θ_2
GMLF	1.2492 (0.4184)	1.8898 (0.3383)	2 (1.3083)	1.6325 (0.0267)	1.1431 (0.0038)	0 (0.0015)
GML2	1.2499 (0.1645)	1.8901 (0.2669)	2 (0.9566)	1.6343 (0.2299)	2.0746 (0.2809)	

Table 7: Metal returns (GARCH residuals): Parameter estimates and corresponding standard errors (in parenthesis)

References

[1] Aas, K.; Czado, C.; Frigessi, A.; Bakken, H. (2006). Pair Copula Constructions of Multiple Dependence, Working Paper of the Norwegian Computing Center.

[2] Berg, D.; Bakken, H. (2006). A Copula Goodness-of-fit Test Based on the Probability Integral Transform, Working Paper of the Norwegian Computing Center.

[3] Breymann, W.; Dias, A.; Embrechts, P. (2003). Dependence structures for multivariate high-frequency data in finance, *Quantitative Finance* 1, 1-14.

[4] Bedford, T.; Cooke, R.M. (2001). Probability density decomposition for conditionally dependent random variables modeled by vines. *Annals of Mathematics and Artificial Intelligence* 32, 245-268.

[5] Bedford, T.; Cooke, R.M. (2002). Vines - a new graphical model for dependent random variables. *Annals of Statistics* 30(4), 1031-1068.

[6] Caputo, A. (1998). Some properties of the family of Koehler Symanowski distributions, The Collaborative Research Center (SBF) 386, working paper No. 103, LMU, München.

[7] Chen, X.; Fan, Y.; Patton, A. (2004). Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates. Working Paper.

[8] Crowson, P.; Sampson, R. (2001). Managing Metal Price Risk with the London Metal Exchange. London Metal Exchange, London.

[9] Embrechts, P.; McNeil, A.; Straumann, D. (1999). Correlation: Pitfalls and Alternatives. *Risk*, 5, 69-71.

[10] Frees, E.W.; Valdez, E.A. (1998). Understanding relationship using copulas. *North American Actuarial Journal*, 2(1), 1-25.

[11] Joe. H. (1996). Families of $m-$variate distributions with given margins and $m(m-1)/2$ bivariate dependence parameters. In L. Rüschendorf and B.Schweizer and M.D.Taylor (Ed.), Distributions with Fixed Marginals and Related Topics.

[12] Joe. H. (1997). Multivariate Models and Dependence Concepts. *Monographs on Statistics and Applied Probability* No. 37, London, Chapman & Hall.

[13] Köhler, K. J.; Symanowski, J. T. (1995). Constructing Multivariate Distributions with Specific Marginal Distributions, *Journal of Multivariate Distributions* 55, 261-282.

[14] Liebscher, E. (2006). Modelling and estimation of multivariate copulas, Working paper, University of Applied Sciences Merseburg.

[15] Morillas, P.M. (2005). A method to obtain new copulas from a given one. *Metrika*, 61, 169-184.

[16] Nelsen, R.B. (2006). An introduction to copulas. Springer Series in Statistics, Springer.

[17] Palmitesta, P.; Provasi, C. (2005). Aggregation of Dependent Risks using the Koehler-Symanowski Copula Function, *Computational Economics* 25, 189-205.

[18] Rosenblatt, M. (1952). Remarks on multivariate Transformation, *The Annals of Statistics* 23, 470-472.

[19] Savu, C.; Trede, M. (2006). Hierarchical Archimedean Copulas, Working Paper, University of Münster.

[20] Sklar, A. (1959). Fonctions de Répartition á n Dimensions et Leurs Marges. *Publications of the Institute of Statistics*, 8, 229-231.

[21] Wang, S. (1998). Aggregation of correlated risk portfolios: models and algorithms, Working paper, www.casact.org.

[22] Whelan, N. (2004). Sampling from Archimedean copulas, *Quantitative Finance* 4, 339-352.