# SOME RESULTS ON WEAK AND STRONG TAIL DEPENDENCE COEFFICIENTS FOR MEANS OF COPULAS

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SUMMARY

Copulas represent the dependence structure of multivariate distributions in a natural way. In order to generate new copulas from given ones, several proposals found its way into statistical literature. One simple approach is to consider convex-combinations (i.e. weighted arithmetic means) of two or more copulas. Similarly, one might consider weighted geometric means. Consider, for instance, the Spearman copula, defined as the geometric mean of the maximum and the independence copula. In general, it is not known whether weighted geometric means of copulas produce copulas, again. However, applying a recent result of Liebscher (2006), we show that every weighted geometric mean of extreme-value copulas produces again an extreme-value copula. The second contribution of this paper is to calculate extremal dependence measures (e.g. weak and strong tail dependence coefficients) for (weighted) geometric and arithmetic means of two copulas.

*Keywords and phrases:* Tail Dependence; Extreme-value copulas; arithmetic and geometric mean

## 1 Copulas and Tail dependence

Let $X$ and $Y$ denote two random variables with joint distribution $F_{X,Y}(x,y)$ and continuous marginal distribution functions $F_X(x)$ and $F_Y(y)$. According to Sklar's (1959) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the *copula*

$$C(u,v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y)$$

defined on $[0,1] \times [0,1]$ which comprises the information about the underlying dependence structure. Putting a different way, (2-dimensional) copulas are distribution functions on the unit square with uniform marginals. For details on copulas we refer to Joe (1999).

The concept of *tail dependence* provides, roughly speaking, a measure for extreme co-movements in the lower and upper tail of $F_{X,Y}(x,y)$, respectively. The upper tail dependence coefficient (TDC) is usually defined by

$$\lambda_U \equiv \lim_{u\to 1^-} P(Y > F_Y^{-1}(u)|X > F_X^{-1}(u)) = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} \in [0,1]. \tag{1.1}$$

noting that $\lambda_U$ is solely depending on $C(u,v)$ and not on the marginal distributions. Analogously, the lower TDC is defined as

$$\lambda_L \equiv \lim_{u\to 0^+} P(Y \le F_Y^{-1}(u)|X \le F_X^{-1}(u)) = \lim_{u\to 0^+} \frac{C(u,u)}{u}. \tag{1.2}$$

Coles et al. (1999) provide asymptotically equivalent versions of (1.1) and (1.2),

$$\lambda_L = 2 - \lim_{u\to 0^+} \frac{\log(1-2u+C(u,u))}{\log(1-u)} \quad \text{and} \quad \lambda_U = 2 - \lim_{u\to 1^-} \frac{\log C(u,u)}{\log(u)}. \tag{1.3}$$

In general, there are a lot of copulas (e.g. Gaussian copula, hyperbolic copula, FGM copula) which admit upper and/or lower tail independence but nevertheless allow a certain dependence between the variables $U$ and $V$ in the tail areas. A measure to quantify "dependence within tail independence" is suggested by Coles et al. (1999) who defines the *weak* upper tail dependence coefficient as

$$\chi_U = \lim_{u\to 1} \chi_U(u) \quad \text{with} \quad \chi_U(u) = \left(\frac{2\log(1-u)}{\log(1-2u+C(u,u))} - 1\right) \text{ for } u \in [0,1],$$

provided the existence. It can be shown that $-1 \le \chi_U \le 1$, $\chi_U = 1$ in case of upper tail dependence (i.e. for $\lambda_U > 0$), $\chi_U = 0$ in case of $C = \Pi$ being the independence copula and for copulas with upper tail independence (i.e. with $\lambda_U = 0$), $\chi_U$ increases with the strength of dependence in the tail area. In the sequel, we speak of *weak upper tail independence* if $\chi_U = 0$, and of *weak upper tail dependence* if $\chi_U \ne 0$. It should be again pointed out that it is not necessary to calculate $\chi_U$ in case of strong upper tail dependence, because then $\chi_U = 1$ holds. Heffernan (2000) calculated $\chi_U$ for numerous copulas. Her derivations require different results from extreme value theory: Because of

$$P(U > u, V > u) = 1 - 2u + C(u,u) = 1 - 2u + P(\max(U,V) \le u)$$

the joint exceedance probability is solely determined by the distribution function of the maximum of $U$ and $V$. Instead of analyzing the limit behaviour for $u \to 1$, one usually considers the bivariate transformation $S = -1/\log U$ and $T = -1/\log V$. The variables $S$ and $T$ have so-called *uniform Fréchet marginal distributions* with

$$P(S > s) = P(T > s) = P(U > e^{-1/s}) = 1 - e^{-1/s} \text{ for } s > 0.$$

Applying a Taylor approximation for large $s$, $e^{-1/s} \approx 1 - \frac{1}{s}$ and $P(S > s) = P(T > s) \approx \frac{1}{s}$. Ledford & Tawn (1996) showed that for uniform Fréchet marginal distributions and under weak conditions

$$P(S > t, T > t) \;\approx\; \mathcal{L}(t) P(S > t)^{1/\eta} \text{ for large } t$$

holds, where $\mathcal{L}(t)$ is a slowly varying function in $\infty$, i.e. with $\frac{\mathcal{L}(ct)}{\mathcal{L}(t)} \to 1$ for $t \to \infty$ for each $c > 0$. Moreover, the coefficient $\eta$ quantifies a weak upper tail dependence coefficient because $\chi_U = 2\eta - 1$. Furthermore, it can be shown that $\lambda_U = c$ in case of $\mathcal{L}(t) \to c$ and $\chi_U = 1$ and $\lambda_U = 0$ in case of $\chi_U < 1$. Moreover, using

$$P(S > t, T > t) = P(U > e^{-1/t}, V > e^{-1/t}) = 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t})$$

the relation between the uniform Fréchet marginal distributions and the copula $C$ can be established. Thus one has to check if there is a function $\mathcal{L}(t)$ slowly varying in $\infty$ and a $\eta$ satisfying

$$1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \;\approx\; \mathcal{L}(t)\left(\frac{1}{t}\right)^{1/\eta} \quad \text{for large } t.$$

Likewise, the weak lower tail dependence coefficient equals the limit of

$$\chi_L(u) = \frac{2\log(u)}{\log(C(u,u))} - 1$$

for $u \to 0$. This limit in turn can be calculated by applying extreme value theory. Again, one has to consider a factorisation of the form

$$C(1 - e^{-1/t}, 1 - e^{-1/t}) \;\approx\; \mathcal{L}(t)\left(\frac{1}{t}\right)^{1/\eta} \quad \text{for large } t.$$

The relationship between $\mathcal{L}(t)$, $\eta$ on the one side and $\lambda_L$, $\overline{\chi}_L$ on the other side is identical to the case of upper tail dependence. For a more detailed treatment of weak and strong tail dependence we refer to Heffernan (2000) and Klein & Fischer (2006).

## 2 The weak TDC of the arithmetic mean of two copulas

In general, every convex-combination of two (or more) copulas $C_1$ and $C_2$

$$C(u, v; \alpha) = \alpha C_1(u, v) + (1 - \alpha)C_2(u, v), \tag{2.1}$$

is again a copula. Convex-combinations are popular to construct flexible copula models (see, e.g. Junker & May, 2005). If the upper TDC of $C_1$ and $C_2$ is given by $\lambda_{U,1}$ and $\lambda_{U,2}$, respectively, it is straightforward to check that the upper TDC $\lambda_U$ of $C$ is given by

$$\lambda_U = \alpha\lambda_{U,1} + (1 - \alpha)\lambda_{U,2}.$$

To the best of our knowledge, there is no result for the *weak* upper tail dependence coefficient $\chi_U$. The corresponding derivation is subject to the next theorem.

**Theorem 1.** *Assume that the weak upper TDC of $C_1$ and $C_2$ is given by $\chi_{U,1}$ and $\chi_{U,2}$. The weak upper TDC $\chi_U$ of the convex combination (i.e. arithmetic mean) of two copulas does not dependence on $\alpha$ and is given by*

$$\chi_U = \max\Big\{\chi_{U,1}, \chi_{U,2}\Big\}.$$

*Proof*: Recall that if

$$1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \approx \mathcal{L}(t) \left(\frac{1}{t}\right)^{1/\eta}$$

holds for large $t$, the weak lower tail dependence coefficient is given by $\chi_U = 2\eta - 1$. Consequently, from the assumptions on $C_1$ and $C_2$ above and for large $t$ follows that

$$\begin{aligned} 1 - 2e^{-1/t} + C_1(e^{-1/t}, e^{-1/t}) &\approx \mathcal{L}_1(t) t^{-1/\eta_1} \quad \text{and} \\ 1 - 2e^{-1/t} + C_2(e^{-1/t}, e^{-1/t}) &\approx \mathcal{L}_2(t) t^{-1/\eta_2}. \end{aligned}$$

where $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$ are slowly varying functions in $\infty$. Hence, for large $t$,

$$\begin{aligned} & 1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) = 1 - 2e^{-1/t} + \alpha C_1(e^{-1/t}, e^{-1/t}) + (1-\alpha) C_2(e^{-1/t}, e^{-1/t}) \\ = \; & \alpha \Big[1 - 2e^{-1/t} + C_1(e^{-1/t}, e^{-1/t})\Big] + (1-\alpha) \Big[1 - 2e^{-1/t} + C_2(e^{-1/t}, e^{-1/t})\Big] \\ \approx \; & \alpha \mathcal{L}_1(t) \left(\frac{1}{t}\right)^{1/\eta_1} + (1-\alpha) \mathcal{L}_2(t) \left(\frac{1}{t}\right)^{1/\eta_2} \end{aligned}$$

Assume w.l.o.g. that $0 < \eta_1 < \eta_2 \leq 1$ such that $1/\eta_1 > 1/\eta_2 > 0$ and $t^{-1/\eta_1} < t^{-1/\eta_2}$ for large $t$ (note that in particular $1/t < 1$). Then, for large $t$,

$$1 - 2e^{-1/t} + C(e^{-1/t}, e^{-1/t}) \approx \mathcal{L}^*(t) \left(\frac{1}{t}\right)^{1/\eta_2}$$

with $\mathcal{L}^*(t) \equiv (1-\alpha)\mathcal{L}_2(t)$. Therefore, the weak tail dependence coefficient is given by $\chi_U = 2\eta_2 - 1$. Otherwise, i.e for $0 < \eta_2 < \eta_1 \leq 1$, the result is $\chi_U = 2\eta_1 - 1$. Consequently, $\chi_U = 2\max(\eta_1, \eta_2) - 1 = \max\{\chi_{U,1}, \chi_{U,2}\}$ holds in the general case. $\square$

# 3 TDC for the geometric mean of two extreme-value copulas

In contrast to the arithmetic mean of two copulas, it is in general not known (as far as we know) if the geometric mean of two arbitrary copulas is again a copula. There is, however, already a result for a specific case, i.e. copula family B12 in Joe (1999) – also known as Spearman or Cuadras-Augé copula – which results as the geometric mean of the maximum copula and the independence copula, i.e.

$$C_G(u, v; \alpha) = \min\{u, v\}^{\alpha} (uv)^{1-\alpha}, \qquad \alpha \in [0, 1] \tag{3.1}$$

Applying a recent result from Liebscher (2006, Theorem 3.1), we are able to prove in the next lemma that the geometric mean of two extrem-value copulas is again an extrem-value copula.

**Lemma 3.1.** *Assume that $C_1$ and $C_2$ are two extreme-value copulas. It follows that*

$$C_G(u,v;\alpha) = C_1(u,v)^\alpha C_2(u,v)^{1-\alpha}, \qquad \alpha \in [0,1] \tag{3.2}$$

*is again an extreme-value copula.*

*Proof*: Using that $C(u^t,v^t) = C(u,v)^t$ for $t > 0$ holds for any extreme-value copula, (3.2) can be rewritten – defining $g_1(u) \equiv u^\alpha$ and $g_2(u) \equiv u^{1-\alpha}$ – as

$$\begin{aligned} C_G(u,v;\alpha) &= C_1(u,v)^\alpha C_2(u,v)^{1-\alpha} = C_1(u^\alpha, v^\alpha) C_2(u^{1-\alpha}, v^{1-\alpha}) \\ &= C_1(g_1(u), g_1(v)) C_2(g_2(u), g_2(v)) \end{aligned}$$

Obviously, $g_1$ and $g_2$ are bijective and monotone increasing functions with $g_1(u) \cdot g_2(u) = u^\alpha u^{1-\alpha} = u$. Hence, applying theorem 3.1 of Liebscher (2006), the assertion follows. Moreover, $C_G(u,v;\alpha)$ is again an extreme-value copula because

$$C_G(u^t, v^t; \alpha) = \left(C_1(u,v)^\alpha C_2(u,v)^{1-\alpha}\right)^t = \left(C_G(u,v;\alpha)\right)^t. \quad \Box$$

The next lemma states that the (upper) tail dependence coefficient $\lambda_U$ of the geometric mean of two extreme-value copulas derives as convex-combination of the TDC of the two copulas itself.

**Lemma 3.2** (Strong TDC)**.** *Assume that $C_1$ and $C_2$ are two extreme-value copulas with upper strong TDC $\lambda_{U,i}$ for $i = 1,2$. Then the TDC of the weighted geometric mean of $C_1$ and $C_2$ is given by*

$$\lambda_U = \alpha\lambda_{U,1} + (1-\alpha)\lambda_{U,2}$$

*Proof*: Plugging the copula from (3.2) into equation (1.3) we obtain

$$\begin{aligned} \lambda_U &= 2 - \lim_{u\to 1} \frac{\log(C_1(u,u)^\alpha C_2(u,u)^{1-\alpha})}{\log(u)} \\ &= 2 - \alpha \lim_{u\to 1} \frac{\log C_1(u,u)}{\log(u)} - (1-\alpha) \lim_{u\to 1} \frac{\log C_2(u,u)}{\log(u)} \\ &= \alpha\Big(2 - \lim_{u\to 1} \frac{\log C_1(u,u)}{\log(u)}\Big) + (1-\alpha)\Big(2 - \lim_{u\to 1} \frac{\log C_2(u,u)}{\log(u)}\Big) \\ &= \alpha\lambda_{U,1} + (1-\alpha)\lambda_{U,2} \quad \Box. \end{aligned}$$

Finally, we turn to the weak tail dependence coefficient. First recall (e.g. Heffernan, 2000) that every extreme-value copula $C$ may be characterized by a so-called *dependence function* $V$ via

$$C(u,v) = \exp(-V(-1/\ln(u), -1/\ln(v))),$$

where exakt independence corresponds to $V(1,1) = 2$ and perfect dependence to $V(1,1) = 1$. Regarding the weak upper TDC, $\eta = 1$ and hence $\chi_U = 1$ for all extreme-value distributions with $V(1,1) \neq 2$. Hence, emphasize is put on the weak lower TDC which can be expressed – provided that $V(1,1) > 1$ – as

$$\chi_L = \frac{2 - V(1,1)}{V(1,1)}. \tag{3.3}$$

This result facilitates the derivation of the weak lower TDC, as the next lemma shows.

**Lemma 3.3** (Weak TDC)**.** *Assume that $C_1$ and $C_2$ are two extreme-value copulas with weak upper TDC $\chi_{L,1}$ and $\chi_{L,2}$, and dependence functions $V_1$ and $V_2$, respectively. Then the TDC of the weighted geometric mean of $C_1$ and $C_2$ is given by is given by*

$$\chi_L = \frac{2 - \alpha V_1(1,1) - (1-\alpha)V_2(1,1)}{\alpha V_1(1,1) + (1-\alpha)V_2(1,1)}$$

*Proof*: Some reformulations reveal the dependence function of $C_G$ which is given as the convex-combination of the dependence function of $C_1$ and $C_2$:

$$\begin{aligned} C_G(u,v;\alpha) &= C_1(u,v)^{\alpha} C_2(u,v)^{1-\alpha} \\ &= \exp\left(-\alpha V_1\left(-\frac{1}{\ln(u)}, -\frac{1}{\ln(v)}\right)\right) \exp\left(-(1-\alpha)V_2\left(-\frac{1}{\ln(u)}, -\frac{1}{\ln(v)}\right)\right) \\ &= \exp\left(-V\left(-\frac{1}{\ln(u)}, -\frac{1}{\ln(v)}\right)\right) \end{aligned}$$

with $V(u,v) \equiv \alpha V_1(u,v) + (1-\alpha)V_2(u,v)$. Consequently, using (3.3), the assertion follows. $\square$

**Example 3.1.** Consider the copula $C_1(u,v)$ which is characterized by the "logistic" dependence function

$$V_1(x,y;\beta) = \left(x^{-1/\beta} + y^{-1/\beta}\right)^{\beta}, \quad 0 < \beta \leq 1$$

with weak lower TDC given by $\chi_{L,1} = 2^{1-\beta} - 1$. Further, let $C_2(u,v) = uv$ with dependence function $V_2(x,y) = V_1(x,y;1) = x^{-1} + y^{-1}$, hence $V_2(1,1) = 2$ and $\chi_{L,2} = 0$. It follows that

$$\chi_L = \frac{2 - \alpha 2^{\beta} - (1-\alpha)2}{\alpha 2^{\beta} + (1-\alpha)2} = \frac{\alpha(2 - 2^{\beta})}{2 + \alpha(2^{\beta} - 2)} \quad \text{and} \quad \chi_U = 1.$$

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