A NEW CLASS OF COPULAS WITH TAIL DEPENDENCE AND A GENERALIZED TAIL DEPENDENCE ESTIMATOR

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SUMMARY

We present a new family of copulas ("generalized mean copulas") which is positive comprehensive and allows for upper tail dependence. It includes the Spearman copula and a specific Fréchet copula as special cases. Some properties and a generalized tail dependence estimator are derived. Finally, a small simulation study is conducted.

Keywords and phrases: Geometric mean; arithmetic mean; copula; tail dependence

1 Introduction

Since the pioneering work of Embrechts et al. (1999) and the research group of Credit Lyonais (e.g. Bouyé et al., 2000), the popularity of the copula concept in finance steadily increases. As it was demonstrated by Sklar (1959), each multivariate probability distribution can be decomposed into its margins and its dependence structure, e.g. by its copula. In contrast to the modelling of the marginal distribution, capturing the adequate dependence structure between the assets under consideration is still a challenging task. Above that, risk managers are especially faced with the problem that assets tend to collapse together, a phenomenon titled as "tail dependence" in the literature. This gives rise to two issues. Firstly, the construction of copulas which are able to assign sufficient probability to these extreme events of common rises or falls. Secondly, the development of simple and accurate tail dependence estimators (TDE) in order to check whether tail dependence is present in a data set, at all. Based on a specific copula which is introduced within this work, we are able to give some contribution to each of these aspects.

In detail, the proceeding is as follows. Section 2 briefly reviews the copula concept and the notion of tail dependence. In section 3, the generalized mean copulas are introduced and some properties are derived. The main focus of section 4 is on a special case, the so-called harmonic mean copulas. Finally, section 5 is dedicated to the development of a family of non-parametric tail dependence estimator and a small simulation study.

2 Copulas: A review

Let $[a, b] \subseteq \mathbb{R}$. A function $K : [a, b] \times [a, b] \to \mathbb{R}$ is said to be 2-*increasing* if its K-volume

$$V_K(u_1, u_2, v_1, v_2) \equiv K(u_2, v_2) - K(u_2, v_1) - K(u_1, v_2) + K(u_1, v_1) \geq 0 \qquad (2.1)$$

for all $a \leq u_1 \leq u_2 \leq b$ and $a \leq v_1 \leq v_2 \leq b$. If, additionally, $[a, b] = [0, 1]$ and K satisfies the boundary conditions

$$K(u, 0) = K(0, v) = 0, \quad K(u, 1) = u \text{ and } K(1, v) = v \qquad (2.2)$$

for arbitrary $u, v \in [0, 1]$, K is commonly termed as copula and we write C, instead.

Putting a different way, let X and Y denote two random variables with joint distribution $F_{X,Y}(x, y)$ and continuous marginal distribution functions $F_X(x)$ and $F_Y(y)$. According to Sklar's (1959) fundamental theorem, there exists a unique decomposition

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y))$$

of the joint distribution into its marginal distribution functions and the so-called copula

$$C(u, v) = P(U \leq u, V \leq v), \quad U \equiv F_X(X), \quad V \equiv F_Y(Y)$$

on $[0, 1]^2$ which comprises the information about the underlying dependence structure (For details on copulas we refer to Nelsen, 2006 and Joe, 1999).

Prominent examples are the *independence* copula

$$C_I(u, v) = uv \qquad (2.3)$$

which corresponds to bivariate distributions with independent marginals and the maximum copula

$$C_U(u, v) = \min\{u, v\}, \qquad (2.4)$$

associated to random variables which are co-monotone and, thus, constituting an upper bound for all copulas. Copulas which include both independent copula (i.e. no dependence) and maximum copula (i.e. perfect dependence) will be termed as *positive comprehensive*, henceforth. Examples are given in the next section.

3 A generalized mean copula

In general, every convex-combination of two (or more) copulas is again a copula. For instance, convex-combining (2.3) and (2.4), family B11 in Joe (1999) is obtained, i.e.

$$C_A(u, v; \alpha) = \alpha \min\{u, v\} + (1 - \alpha)uv, \qquad \alpha \in [0, 1] \qquad (3.1)$$

which is also a special case of the Fréchet family (Fréchet, 1958). In other words, C_A is a (weighted) arithmetic mean of C_I and C_U and termed as arithmetic mean copula, henceforth. Similarly, family B12 in Joe (1999) – also Spearman or Cuadras-Augé copula – is given by

$$C_G(u, v; \alpha) = \min\{u, v\}^{\alpha}(uv)^{1-\alpha}, \qquad \alpha \in [0, 1] \tag{3.2}$$

which can be seen as a weighted geometric mean of C_I and C_U. In both cases, $\alpha = 0$ results in the independence, whereas $\alpha = 1$ results in the maximum copula. Hence, both copula families are positive comprehensive. A natural generalization of (3.1) and (3.2) is given by a weighted Hölder (or power) mean of C_I and C_U. This generalized mean, also known as Hölder mean, is an abstraction of the (weighted) arithmetic and (weighted) geometric means (see, e.g. Borwein & Borwein, 1987 or Bullen, 2003). For $m \in \mathbb{R}$ and $\alpha \in [0, 1]$, consider

$$K(u, v; \alpha, m) \equiv (\alpha \min\{u, v\}^m + (1 - \alpha)(uv)^m)^{1/m} . \tag{3.3}$$

Note that $K(u, v; \alpha, 1) = C_A(u, v)$ and – after taking the limit – $K(u, v; \alpha, 0) = C_G(u, v)$. We next show that K is a copula for $m \in \mathbb{R} \setminus \{0, 1\}$, too. The boundary conditions are easily verified: $K(u, 0; \alpha, m) = K(0, v; \alpha, m) = 0$, $K(u, 1; \alpha, m) = u$ and $K(1, u; \alpha, m) = v$ for all $u, v \in [0, 1]$. Moreover, K is an exchangeable function, i.e. $K(u, v; \alpha, m) = K(v, u; \alpha, m)$. In order to proof that K is actually a copula it remains to verify that K satisfies the two-increasing condition from (2.1) with

$$\begin{aligned} V_K \;=\; & (\alpha \min\{u_2, v_2\}^m + (1 - \alpha)(u_2 v_2)^m)^{\frac{1}{m}} - (\alpha \min\{u_2, v_1\}^m + (1 - \alpha)(u_2 v_1)^m)^{\frac{1}{m}} \\ & - (\alpha \min\{u_1, v_2\}^m + (1 - \alpha)(u_1 v_2)^m)^{\frac{1}{m}} + (\alpha \min\{u_1, v_1\}^m + (1 - \alpha)(u_1 v_1)^m)^{\frac{1}{m}} . \end{aligned}$$

Lemma 3.1. *$K(u, v; \alpha, m)$ from (3.3) is a copula for $\alpha \in [0, 1]$ and $m \in \mathbb{R}$.*

Proof: In order to proof that $V_K \geq 0$, consider the following cases

- **Case 1**: $u_1 \leq v_1 \leq u_2 \leq v_2$,
- **Case 2**: $u_1 \leq v_1 \leq v_2 \leq u_2$,
- **Case 3**: $u_1 \leq u_2 \leq v_1 \leq v_2$,
- **Case 4**: $v_1 \leq u_1 \leq v_2 \leq u_2$,
- **Case 5**: $v_1 \leq u_1 \leq u_2 \leq v_2$,
- **Case 6**: $v_1 \leq v_2 \leq u_1 \leq u_2$.

Introducing the auxiliary function

$$f(u; \alpha, m) \equiv (\alpha + (1 - \alpha)u^m)^{1/m}$$

we have to check the validity of the following inequalities:

- **Case 1**: $u_2 f(v_2) - v_1 f(u_2) - u_1 f(v_2) + u_1 f(v_1) \geq 0$,
- **Case 2**: $v_2 f(u_2) - v_1 f(u_2) - u_1 f(v_2) + u_1 f(v_1) = (v_2 - v_1) f(u_2) - u_1 (f(v_2) - f(v_1)) \geq 0$,
- **Case 3**: $u_2 f(v_2) - u_2 f(v_1) - u_1 f(v_2) + u_1 f(v_1) = (u_2 - u_1)(f(v_2) - f(v_1)) \geq 0$,
- **Case 4**: $v_2 f(u_2) - v_1 f(u_2) - u_1 f(v_2) + v_1 f(u_1) \geq 0$,
- **Case 5**: $u_2 f(v_2) - v_1 f(u_2) - u_1 f(v_2) + v_1 f(u_1) = (u_2 - u_1) f(v_2) - v_1 (f(u_2) - f(u_1)) \geq 0$,
- **Case 6**: $v_2 f(u_2) - v_1 f(u_2) - v_2 f(u_1) + v_1 f(u_1) = (v_2 - v_1)(f(u_2) - f(u_1)) \geq 0$.

Actually, using the exchangeability of K, it suffices to prove case 1, case 2 and case 3.

Case 1: Assume that $u_1 \leq v_1 \leq u_2 \leq v_2$. Due to lemma 3.2(5.), $v_1 f(u_2) \leq u_2 f(v_1)$ and it follows that

$$u_2 f(v_2) - v_1 f(u_2) - u_1 f(v_2) + u_1 f(v_1) \geq u_2 f(v_2) - u_2 f(v_1) - u_1 f(v_2) + u_1 f(v_1)$$

and $u_2 f(v_2) - u_2 f(v_1) - u_1 f(v_2) + u_1 f(v_1) = (u_2 - u_1)(f(v_2) - f(v_1)) \geq 0$.
Case 2: Assuming that $u_1 \leq v_1 \leq v_2 \leq u_2$ we have to show that

$$(v_2 - v_1) f(u_2) - u_1 (f(v_2) - f(v_1)) \geq 0.$$

We restrict ourselves to $0 < v_1 < v_2$ because $v_1 = v_2$ and $u_1 = v_1 = 0$ is trivial. Now $f(u_2) \geq f(v_2)$ (cf. lemma 3.2(1.)) and $u_1 \leq v_1$ and therefore

$$(v_2 - v_1) f(u_2) - u_1 (f(v_2) - f(v_1)) \geq (v_2 - v_1) f(v_2) - v_1 (f(v_2) - f(v_1))$$

and it suffices to show that

$$(v_2 - v_1) f(v_2) - v_1 (f(v_2) - f(v_1)) \geq 0 \quad \Longleftrightarrow \quad \frac{f(v_2)}{v_1} \geq \frac{f(v_2) - f(v_1)}{v_2 - v_1}.$$

Denoting $v_2 = v_1 + \triangle$, we can rewrite the last inequality to

$$\frac{f(v_1 + \triangle)}{v_1} \geq \frac{f(v_1 + \triangle) - f(v_1)}{\triangle}.$$

Now letting $\triangle \to 0$, we have to show that

$$\frac{f(v_1)}{v_1} = \lim_{\triangle \to 0} \frac{f(v_1 + \triangle)}{v_1} \geq \lim_{\triangle \to 0} \frac{f(v_1 + \triangle) - f(v_1)}{\triangle} = f'(v_1).$$

This, however follows from lemma 3.1(4).

Case 3: Under the above assumption and with lemma 3.2(1) – where it is established that f is monotone increasing – the assertion follows immediately. $\square$

Lemma 3.2. *Let* $\alpha \in [0,1]$ *and* $m \in \mathbb{R}$. *Then*

1. f *is strictly monotone increasing for* $u \in [0,1]$.
2. $f(0) = \alpha^{1/m}$ *for* $m > 0$ *and, after taking limits,* $f(0) = 0$ *for* $m < 0$. *Furthermore,* $f(1) = 1$.
3. *It is for* $u \in [0,1]$
$$u \leq (\alpha + (1-\alpha)u^m)^{1/m} \leq 1.$$
4. *For* $u \in [0,1]$ *we have* $f(u) \geq uf'(u)$.
5. *For* $\triangle \geq 0$ *we have* $uf(u+\triangle) \leq (u+\triangle)f(u)$.
6. f *is concave for* $m > 1$, *convex for* $m < 1$ *and linear for* $m = 1$.

Proof:

1. Follows from $f'(u) = (1-\alpha)\left(\alpha + (1-\alpha)u^m\right)^{1/m-1} u^{m-1} \geq 0$.
2. Obvious.
3. The upper bound follows from 1. and 2. For $m > 0$, the lower bound holds if $\alpha(1 - u^m) \geq 0$ which is valid for $\alpha \geq 0$ and $0 \leq u \leq 1$. For $m < 0$, the lower bound holds if $\alpha(1-u^m) \leq 0$ which is valid for $\alpha \geq 0$ and $0 \leq u \leq 1$.
4. From the equivalence
$$\begin{aligned} (\alpha + (1-\alpha)u^m)^{1/m} &\geq u(1-\alpha)\left(\alpha + (1-\alpha)u^m\right)^{1/m-1} u^{m-1} \iff \\ (\alpha + (1-\alpha)u^m)^{1/m} &\geq (1-\alpha)\left(\alpha + (1-\alpha)u^m\right)^{1/m-1} u^{m} \iff \\ \alpha + (1-\alpha)u^m &\geq (1-\alpha)u^m \iff \alpha \geq 0. \end{aligned}$$
5. For $m > 0$, notice the equivalence $uf(u+\triangle) \leq (u+\triangle)f(u) \iff$
$$\begin{aligned} u\left(\alpha + (1-\alpha)(u+\triangle)^m\right)^{1/m} &\leq (u+\triangle)\left(\alpha + (1-\alpha)u^m\right)^{1/m} \iff \\ u^m\left(\alpha + (1-\alpha)(u+\triangle)^m\right) &\leq (u+\triangle)^m\left(\alpha + (1-\alpha)u^m\right) \iff u^m \leq (u+\triangle)^m. \end{aligned}$$
The derivation for $m < 0$ is similar, but now $u^m \geq (u+\triangle)^m$.
6. Follows from
$$f''(u) = \frac{a\,(a-1)\,(a + u^m - u^m a)^{1/m-1}\, u^{m-2}\,(m-1)}{(-a + u^m(a-1))} \lesseqgtr 0 \text{ for } m \gtrless 1. \qquad \square$$

Contour lines of generalized mean copulas are given in figure 1, below.

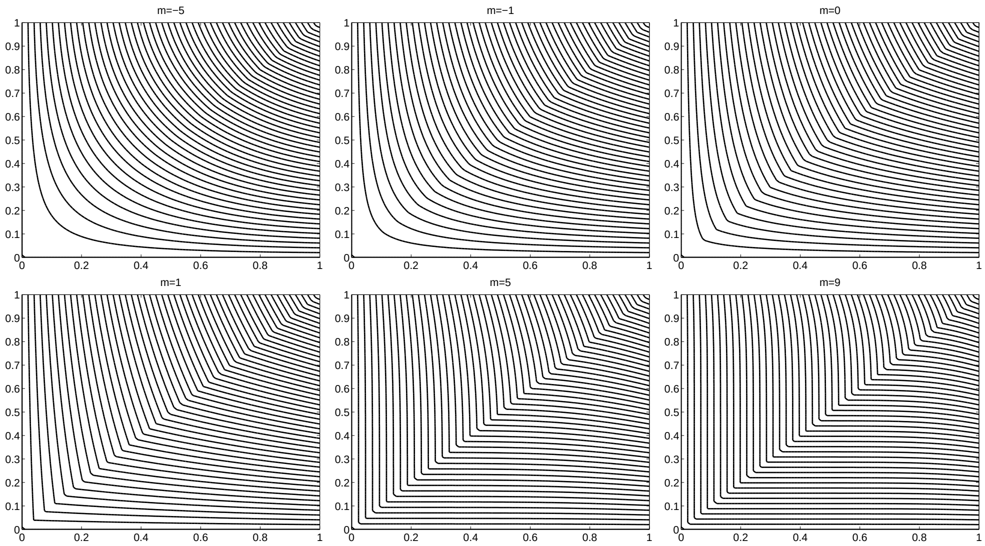

Figure 1: $\alpha = 0.5$

4 A special case: The harmonic mean copula

We now restrict ourselves to a specific member of the generalized mean copula family, namely the so-called *harmonic mean copulas* (HMC) which emerge for $m = -1$, i.e.

$$C_H(u, v; \alpha) \equiv \frac{1}{\frac{\alpha}{\min\{uv\}} + \frac{1-\alpha}{uv}} = \frac{\min\{u, v\} \cdot uv}{\alpha uv + (1-\alpha)\min\{u, v\}} = \frac{C_G(u, v; 0.5)^2}{C_A(u, v; \alpha)}. \tag{4.1}$$

For the HMC-copulas global dependence measures like Kendall's τ, Spearman's ρ, Gini's γ and Blomqist's β can be derived explicitly (For a detailed treatment of dependence measure we refer to Drouet-Mari & Kotz, 2002).

Lemma 4.1. *For the harmonic mean copula $C_H(u, v; \alpha)$ we obtain*

$$\tau = \tau(\alpha) = \frac{18\alpha^2 - 12\alpha - 4\alpha^3 - \alpha^4 + (24\alpha - 12\alpha^2 - 12)\ln(1-\alpha)}{\alpha^4}$$

$$\rho_S = \rho_S(\alpha) = \frac{12\alpha - 30\alpha^2 + 22\alpha^3 - 3\alpha^4 + (12 - 36\alpha + 36\alpha^2 - 12\alpha^3)\ln(1-\alpha)}{\alpha^4},$$

$$\gamma = \gamma(\alpha) = \frac{3\alpha^2 - 2\alpha^3 + \ln(1-\alpha/2)(8-8\alpha) - \ln(1-\alpha)(4 - 8\alpha + 4\alpha^2)}{\alpha^3},$$

$$\beta = \beta(\alpha) = \frac{\alpha}{2-\alpha}.$$

Proof: The main obstacle is to get rid off the minimum expression in the definition of C_H in (4.1). Following Cherubini, Luciano & Vecchiato (2004, Chapter 3.1), Kendall's τ for copulas which are singular or have both an absolutely continuous and a singular component can be computed by

$$\begin{aligned}
\tau &= 1-4\int_0^1\int_0^1 \frac{\partial C(u,v)}{\partial u}\frac{\partial C(u,v)}{\partial v}\,du\,dv \\
&= 1-4\int_0^1\left[\int_0^v \frac{\partial\left(\frac{uv}{\alpha v+1-\alpha}\right)}{\partial u}\frac{\partial\left(\frac{uv}{\alpha v+1-\alpha}\right)}{\partial v}du+\int_v^1 \frac{\partial\left(\frac{uv}{\alpha u+1-\alpha}\right)}{\partial u}\frac{\partial\left(\frac{uv}{\alpha u+1-\alpha}\right)}{\partial v}du\right]dv \\
&= 1-4\int_0^1\left[\int_0^v -\frac{vu\,(-1+\alpha)}{(\alpha\,v+1-\alpha)^3}du-\int_v^1 \frac{vu\,(-1+\alpha)}{(\alpha\,u+1-\alpha)^3}du\right]dv \\
&= 1-4\int_0^1 \frac{v\,(-1+\alpha)\left(v^3\alpha-3\,\alpha\,v-1+2\,\alpha+\alpha^2v^3-3\,\alpha^2v^2+3\,v\alpha^2-\alpha^2\right)}{(\alpha\,v+1-\alpha)^3}dv \\
&= \frac{18\alpha^2-12\alpha-4\alpha^3-\alpha^4+(24\alpha-12\alpha^2-12)\ln(1-\alpha)}{\alpha^4}.
\end{aligned}$$

The result on Spearman's ρ follows with

$$\begin{aligned}
\rho_S &= 12\int_0^1\int_0^1 C_H(u,v;\alpha)dudv-3 \\
&= 12\int_0^1\left(\int_0^v \frac{uv}{\alpha v+1-\alpha}du+\int_v^1 \frac{uv}{\alpha u+(1-\alpha)}du\right)dv-3
\end{aligned}$$

and some tedious but straightforward calculations. Similarly, Gini's γ follows from

$$\begin{aligned}
\gamma &= 4\int_0^1 C(u,1-u)+C(u,u)du-2 \\
&= 4\left(\int_0^{0.5}\frac{u(1-u)}{1-\alpha u}du+\int_{0.5}^1\frac{u(1-u)}{\alpha u+(1-\alpha)}du+\int_0^1\frac{u^2}{\alpha u+(1-\alpha)}\right)du-2.
\end{aligned}$$

Finally, Blomquist's β is simply $\beta = 4C_H(0.5,0.5;\alpha)-1$. □

Figure 2, below illustrates the dependence of τ, ρ and γ from the parameter α. All curves are strictly monotone increasing and convex. The results from last lemma may be useful to estimate the parameters of the harmonic mean copulas.

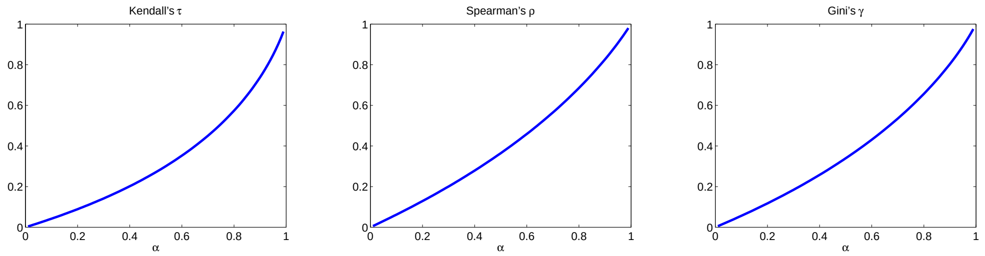

Figure 2: Influence of α on different dependence measures.

5 Derivation of the (upper) tail dependence coefficient and a generalized TDC estimator

The concept of tail dependence provides, roughly speaking, a measure for extreme co-movements in the lower and upper tail of $F_{X,Y}(x,y)$, respectively and is very useful in financial risk management. Regarding the generalized mean copulas, we focus on the upper tail dependence coefficient (TDC) which is usually defined by

$$\lambda_U \equiv \lim_{u\to 1^-} P(Y > F_Y^{-1}(u)|X > F_X^{-1}(u)) = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u} \in [0,1] \tag{5.1}$$

noting that λ_U is solely depending on the copula and not on the marginal distributions. Coles et al. (1999) provide an asymptotically equivalent version of (5.1),

$$\lambda_U = 2 - \lim_{u\to 1^-} \frac{\log C(u,u)}{\log(u)}. \tag{5.2}$$

Fischer & Dörflinger (2006) showed that $C_G(u,v)$ from (3.2) is upper tail dependent with TDC $\lambda_U = \alpha \in [0,1]$. This result also holds for the generalized mean copula, as the next lemma shows.

Lemma 5.1. *The (upper) TDC of the generalized mean copula is given by $\lambda = \alpha$.*

Proof: Plugging (3.3) into (5.2) and applying l'Hospital's rule, we obtain

$$\begin{aligned} \lambda_U &= 2 - \frac{1}{m}\lim_{u\to 1^-} \frac{\log\left(\alpha u^m + (1-\alpha)u^{2m}\right)}{\log(u)} \\ &= 2 - \frac{1}{m}\lim_{u\to 1^-} \frac{u(m\alpha u^{m-1} + 2m(1-\alpha)u^{2m-1})}{\alpha u^m + (1-\alpha)u^{2m}} = 2-(2-\alpha) = \alpha. \quad \square \end{aligned}$$

Hence, the upper TDC λ_U is solely determined by the parameter α and not by m. This allows to construct a generalized TDC-estimator which includes the Dobric-Schmid (2005) estimator for $m = 1$ and the Fischer-Dörflinger (2006) estimator for $m = 0$ as special case. The main idea is simply as follows: At first, approximate the unknown "data-generating" copula $C(u, v)$ by the generalized mean copula $C(u, v; \alpha, m)$. Secondly, choose α and m such that the squared difference between the empirical copula C_n and the generalized mean copula is minimized. For practical purposes, $\mathcal{M}$ is assumed to be a discrete subset of $\mathbb{R}$. Noting that λ_U of the latter is given by α, choose $\widehat{\lambda}_U$ as solution of

$$\widehat{\lambda}_U = \widehat{\alpha} = \min_{m \in \mathcal{M}} \widehat{\alpha}(m) \quad \text{with} \quad \widehat{\alpha}(m) = \text{argmin}_\alpha f(\alpha),$$

$$f(\alpha) \equiv \sum_{i=1}^{k} \left(C_n \left(1 - \frac{i}{n}, 1 - \frac{i}{n} \right) - C \left(1 - \frac{i}{n}, 1 - \frac{i}{n}; \alpha, m \right) \right)^2 .$$

As usually, the empirical (Deheuvel) copula is defined by

$$C_n(i/n, j/n) = \frac{1}{n} \sum_{l=1}^{n} \mathbf{1}(X_l \leq X_{(i)}, Y_l \leq Y_{(j)}), \tag{5.3}$$

where n denotes the number of data pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ and $X_{(i)}, Y_{(i)}$ the corresponding order statistics.

Finally, we conducted a small simulation study. In order to compare the quality of the new tail dependence estimators, we simulated from a bivariate Student-t copula with 3 degrees of freedom und $\rho = 0.2$ (Scenario A). In this case, the theoretical upper tail dependence coefficient is approximately 0.178 (for the exact formula we refer to Dobric & Schmid, 2005). In scenario B, random pairs from a rotated Clayton copula with dependence parameter $\theta = 0.5$ are considered. In this case, the theoretical upper TDC is $2^{-1/\theta} = 0.25$. In each case, $n = 2000$ random pairs were repeatedly drawn (with $N = 1000$ repetitions). The corresponding box plots are subject to figure 3.

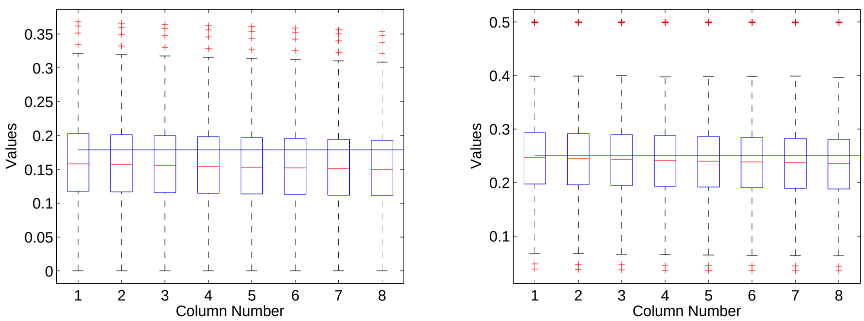

Figure 3: Estimation results for the tail dependence estimators (Box plots).

Note that the column numbers 1 to 8 belong to the estimators with $m = -5$ to $m = 2$. The drawn through line equals the theoretical (true) TDC. From scenario A it becomes obvious that all TDC-estimator underestimate the true tail dependence parameter. However, the bias becomes smaller as m declines. Regarding scenario B, the smaller m the less the bias of the corresponding TDC-estimator.

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