TESTING FOR CONSTANT CORRELATION BY MEANS OF TRIGONOMETRIC FUNCTIONS

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Key Words: Lagrange multiplier test, constant correlation, trigonometric functions.

ABSTRACT

A new test for constant correlation is proposed. The TC-test is derived as Lagrange multiplier (LM) test. Whereas most of the traditional tests (e.g. Jennrich, 1970, Tang, 1995 and Goetzmann, Li & Rouwenhorst, 2005) specify the unknown correlations as piecewise constant, our model-setup for the correlation coefficient is based on trigonometric functions. The simulation results demonstrate that the TC-test guarantees correct empirical size, is powerful against many alternatives and able to detect structural breaks in correlations. Finally, application of the TC-test to foreign exchange rate data over the period of 15 years is given.

1. INTRODUCTION

The classical concept of linear correlations dates back to 1885. Even though there are more powerful measures of dependence (e.g. Mari & Kotz, 2001) correlation coefficients still dominate both theoretical models and practical applications. Ignoring the discussion about the adequate dependence measure and agreeing on correlations henceforth, it still arises the question whether correlations vary in time or not. Particularly the increasing linkages of countries, firms and markets foreshadow a rising correlation of economic and financial time series. Nevertheless, the literature on statistical tests for constant (unconditional) correlations – even for lower dimensions – seems to be comparatively sparse.

It was R. A. Fisher (1915) who first considered the distribution of the correlation coefficient of a set of independent and bivariate Gaussian variables. Generalizing Bartlett's test on equal variances of k samples, Box (1949) introduced a test for equality of covariance matrices which was discussed by Kullback (1967) and Tang (1995) against the background of correlation matrices. Unfortunately, approximating the distribution of the underlying test statistic is still a critical subject. For that reason, the χ^2-test of Jennrich (1970) was the standard procedure to test for equality of correlation matrices for a long time, though assuming that the underlying observation vectors are independent and normally distributed. Recently, Goetzmann, Li & Rouwenhorst (2005) relax the assumption of normality and derive a χ^2-test which applies to distribution families with finite fourth moments.

All of these tests specify the correlation coefficients as piecewise constant over time and verify whether the constants coincide. In contrast, we propose a Lagrange multiplier-type test which allows the correlation coefficient to vary in time according to certain trigonometric functions. Due to its construction, our test also applies to alternative dependence measures, different distributions families and to alternative functional specifications for the unknown correlation.

The proceeding is as follows. Section 2 briefly summarizes different tests on constant (unconditional) correlation from the relevant literature. In section 3 the TC-test is introduced. Results on size and power are provided in section 4. Section 5 is dedicated to empirical application. Section 6 concludes.

2. TESTING FOR CONSTANT CORRELATIONS: A REVIEW

Though all of the following tests for constant correlation are designed for the multivariate case, we restrict discussion to the bivariate case, henceforth. In general, these tests are rooted on Bartlett's test on equal variances, say σ_1^2 and σ_2^2, of two *iid*-normally distributed random samples with possibly different lengths N_1 and N_2. Denoting the sample variance of group j by S_j^2 and defining a pooled sample variance $S^2 = \sum_{j=1}^2 \frac{N_j-1}{N_1+N_2-2} S_j^2$, Bartlett's test statistic is given by

$$\mathcal{T}_{Bartlett} = (N_1 + N_2 - 2)\ln(S^2) - \sum_{j=1}^{2}(N_j - 1)\ln(S_j^2) \quad \approx \quad \chi^2(1). \tag{1}$$

Box (1949) extended Bartlett's proposal to a test for homogeneity of covariance matrices, say $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$, of two subperiods. Equation (1) generalizes to

$$\mathcal{T}_{Box} = (N_1 + N_2 - 2)\ln(\det(\mathbf{S})) - \sum_{j=1}^{2}(N_j - 1)\ln(\det(\mathbf{S}_j)) \text{ with } \mathbf{S} \equiv \sum_{j=1}^{2}\frac{N_j - 1}{N_1 + N_2 - 2}\mathbf{S}_j,$$

where $\mathbf{S}_j$ denotes the sample covariance matrix of subperiod j. Assuming independent and bivariate normally distributed random samples, Box (1949) proposes both a χ^2- and an $F-$approximation to his test statistic $\mathcal{T}_{Box}$. Finally, Kullback (1967) and Tang (1995) deal with the application of Box's test to correlation matrices rather than covariance matrices (by substituting the covariance matrices by the corresponding correlation matrices in the last formula). In particular, Kullback (1967) asserts that if all populations have the same non-singular correlation matrix, then the distribution of the test statistics is asymptotically chi-squared with certain degrees of freedom. However, Jennrich (1970, p. 905) presented a counterexample where Kullback's assertion fails. Jennrich (1970) itself suggested a test for equality of correlation matrices. Under the assumption of independent samples from two k-variate normal populations, the vector $\mathbf{d}$ – which contains all $k^* = k(k-1)/2$ dissimilar element-by-element differences of the two sample correlation matrices in lexographic order – is asymptotically normal with mean zero and non-singular covariance matrix $\mathbf{\Gamma}$. Therefore,

$$\mathcal{T}_{Jennrich} = \frac{N_1 N_2}{N_1 + N_2} \cdot \mathbf{d}'\widehat{\mathbf{\Gamma}}^{-1}\mathbf{d} \quad \overset{a}{\sim} \quad \chi^2(k^*),$$

where $\widehat{\mathbf{\Gamma}}$ is a consistent estimator of $\mathbf{\Gamma}$. Jennrich's main contribution was to derive a simple representation for the inverse of $\widehat{\mathbf{\Gamma}}$ which also applies to high dimensions in a simple way. In order to get rid off the normality assumption, Goetzmann, Li & Rouwenhorst (2005) utilize the asymptotic distribution of the correlation matrix from Browne & Shapiro (1986) and Neudecker & Wesselman (1990). Their proposal only requires that the observation vectors are independent and identically distributed according to a multivariate distribution with finite fourth moments.

Note that all of these tests presume that correlation is piecewise constant in time. In contrast, the TC-test which is introduced next section allows correlation to vary in time according to certain trigonometric functions.

3. A TRIGONOMETRIC TEST FOR CONSTANT CORRELATIONS

For reasons of clearness we focus on the bivariate case, henceforth. Given T pairs of independent random variables $(X_1, Y_1), \ldots, (X_T, Y_T)$ we assume that $(X_t, Y_t) \sim \mathcal{F}(\boldsymbol{\mu}, \mathbf{\Sigma}_t)$ for $t = 1, \ldots, T$, where $\mathcal{F}$ denotes an arbitrary distribution family with (existing) mean vector $\boldsymbol{\mu} \equiv (\mu_X, \mu_Y)'$ and (existing) covariance matrix

$$\mathbf{\Sigma}_t \equiv \begin{pmatrix} \sigma_X^2 & \rho_t \\ \rho_t & \sigma_Y^2 \end{pmatrix} \qquad \text{with} \qquad \rho_t = \frac{Cov(X_t, Y_t)}{\sigma_X \sigma_Y}.$$

Without loss of generality, $\mathcal{F}$ is supposed to be multivariate Gaussian in the sequel. Most of the traditional tests rest upon the assumption that correlation coefficients are piecewise constant over time, i.e.

$$\rho_t = \rho_i \mathbf{1}_{[t_i, t_{i-1})}(t) \;\; \text{with} \;\; 1 = t_0 \leq \cdots \leq t_k = T, \quad \rho_i \in [-1, 1] \;\; \text{for} \;\; i = 1, \ldots, k.$$

In contrast, we advocate a parametric specification based on trigonometric functions, e.g.

$$\rho_t \equiv \beta_0 + \beta_1 \sin(2f\pi t/T) + \beta_2 \cos(2f\pi t/T), \quad t = 1, \ldots, T \tag{2}$$

with unknown frequency $f \in \mathbb{R}$ and unknown coefficients $\beta_0, \beta_1, \beta_2$ which guarantee that $-1 \leq \rho_t \leq 1$. For a given frequency f, testing the null hypothesis of constant correlation equals testing the null hypothesis $H_0 : \beta_1 = \beta_2 = 0$. For this purpose, a likelihood ratio ($\mathcal{LR}$) test may be applied where the difference between the log likelihood under H_0, say ℓ_0, and the overall log likelihood, say ℓ, is considered:

$$\mathcal{LR}(f) \equiv (-2)(\ell_0 - \ell) \quad \overset{a}{\sim} \quad \chi^2(2).$$

Unfortunately, f remains unknown. Following Becker, Ender & Hurn (2004), one might choose a finite set $\Upsilon = \{f_1, \ldots, f_F\}$ of frequencies and consider the test statistics

$$\mathcal{LR}_{sup} \equiv \sup_{f \in \Upsilon} \mathcal{LR}(f), \quad \mathcal{LR}_{ave} \equiv \frac{1}{F} \sum_{f \in \Upsilon} \mathcal{LR}(f) \text{ and } \mathcal{LR}_{exp} \equiv \log \left(\frac{1}{F} \sum_{f \in \Upsilon} \exp(\mathcal{LR}(f)/2) \right)$$

instead. As all test statistics are non-standard, bootstrap methods are necessary to determine the corresponding critical values: First, J replications of the data (which the same means, variances and covariances) have to be generated. Second, for each bootstrap sample, the corresponding test statistic is calculated. Finally, the proportion of the J bootstrapped test statistics which exceed the test statistic computed from the data is then an estimate of the p-value of the test. However, this procedure requires plenty of (unrestricted) maximum likelihood estimations and one may imagine how computing time explodes. We therefore suggest to use a Lagrange multiplier (LM)-type test rather then a LR-type test: The contribution of observation t to the log likelihood is given by

$$\begin{aligned} \ell_t(\boldsymbol{\theta}) &= -\log(2\pi) - \log(\sigma_1) - \log(\sigma_2) - 0.5\log(1-\rho_t^2) - \frac{1}{2(1-\rho_t^2)} \left(\frac{x_{1t} - \mu_1}{\sigma_1} \right)^2 \\ &\quad - \frac{1}{2(1-\rho_t^2)} \left(\frac{x_{2t} - \mu_2}{\sigma_2} \right)^2 + \frac{\rho_t}{1-\rho_t^2} \left(\frac{x_{1t} - \mu_1}{\sigma_1} \right) \left(\frac{x_{2t} - \mu_2}{\sigma_2} \right) \end{aligned}$$

with first partial derivative – with respect to $\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \beta_0, \beta_1, \beta_2)'$ – given by

$$\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \mu_1} = \frac{x_{1t} - \mu_1}{\sigma_1^2(1-\rho_t^2)} - \frac{\rho_t}{1-\rho_t^2} \frac{x_{2t} - \mu_2}{\sigma_1 \sigma_2}, \quad \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \mu_2} = \frac{x_{2t} - \mu_2}{\sigma_2^2(1-\rho_t^2)} - \frac{\rho_t}{1-\rho_t^2} \frac{x_{1t} - \mu_1}{\sigma_1 \sigma_2},$$

$$\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \sigma_1} = -\frac{1}{\sigma_1} + \frac{(x_{1t} - \mu_1)^2}{\sigma_1^3(1-\rho_t^2)} - \frac{\rho_t}{1-\rho_t^2} \frac{(x_{1t} - \mu_1)(x_{2t} - \mu_2)}{\sigma_1^2 \sigma_2},$$

$$\frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \sigma_2} = -\frac{1}{\sigma_2} + \frac{(x_{2t} - \mu_2)^2}{\sigma_2^3(1-\rho_t^2)} - \frac{\rho_t}{1-\rho_t^2} \frac{(x_{1t} - \mu_1)(x_{2t} - \mu_2)}{\sigma_1 \sigma_2^2},$$

$$\begin{aligned} \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \beta_i} &= \frac{\rho_t}{1-\rho_t^2} \frac{\partial \rho_t}{\partial \beta_i} - \frac{\rho_t}{(1-\rho_t^2)^2} \left(\frac{x_{1t} - \mu_1}{\sigma_1} \right)^2 \frac{\partial \rho_t}{\partial \beta_i} - \frac{\rho_t}{(1-\rho_t^2)^2} \left(\frac{x_{2t} - \mu_2}{\sigma_2} \right)^2 \frac{\partial \rho_t}{\partial \beta_i} \\ &\quad + \frac{1+\rho_t^2}{(1-\rho_t^2)^2} \left(\frac{x_{1t} - \mu_1}{\sigma_1} \right) \left(\frac{x_{2t} - \mu_2}{\sigma_2} \right) \frac{\partial \rho_t}{\partial \beta_i}, \end{aligned}$$

where $\frac{\partial \rho_t}{\partial \beta_0} = 1, \quad \frac{\partial \rho_t}{\partial \beta_1} = \sin(2f\pi t/T)$ and $\frac{\partial \rho_t}{\partial \beta_2} = \cos(2f\pi t/T).$

Defining further the score function

$$\boldsymbol{s}(\boldsymbol{\theta}) \equiv \sum_{t=1}^{T} \left(\frac{\partial \ell_t}{\partial \mu_1}, \frac{\partial \ell_t}{\partial \mu_2}, \frac{\partial \ell_t}{\partial \sigma_1^2}, \frac{\partial \ell_t}{\partial \sigma_2^2}, \frac{\partial \ell_t}{\partial \beta_0}, \frac{\partial \ell_t}{\partial \beta_1}, \frac{\partial \ell_t}{\partial \beta_2} \right)' \text{ and } \boldsymbol{S} \equiv \left\{ s_{ti} \equiv \frac{\partial \ell_t(\boldsymbol{\theta})}{\partial \theta_i} \right\}_{t=1,\dots,T, i=1,\dots,7},$$

the corresponding LM-type test statistics are given by

$$\mathcal{LM}_{sup} \equiv \sup_{f \in \Upsilon} \mathcal{LM}(f), \; \mathcal{LM}_{ave} \equiv \frac{1}{K} \sum_{f \in \Upsilon} \mathcal{LM}(f) \text{ and } \mathcal{LM}_{exp} \equiv \log \left(\frac{1}{K} \sum_{f \in \Upsilon} \exp \left(\frac{\mathcal{LM}(f)}{2} \right) \right)$$

$$\text{with } \; \mathcal{LM}(f) \equiv \widehat{\boldsymbol{s}}' \left(\widehat{\boldsymbol{S}}' \widehat{\boldsymbol{S}} \right)^{-1} \widehat{\boldsymbol{s}} \quad \text{for } \widehat{\boldsymbol{s}} \equiv \boldsymbol{s}(\widehat{\boldsymbol{\theta}}_{ML}), \; \widehat{\boldsymbol{S}} \equiv \boldsymbol{S}(\widehat{\boldsymbol{\theta}}_{ML}).$$

Note that, under normality, the maximum likelihood estimator of $\boldsymbol{\theta}$ (under H_0) equals

$$\widehat{\mu}_X = \frac{1}{T} \sum_{i=1}^{T} x_i, \quad \widehat{\mu}_Y = \frac{1}{T} \sum_{i=1}^{T} y_i, \quad \sigma_X^2 = \frac{1}{T} \sum_{i=1}^{T} (x_i - \overline{x})^2, \quad \sigma_Y^2 = \frac{1}{T} \sum_{i=1}^{T} (y_i - \overline{y})^2$$

and

$$\widehat{\rho} = \left(\frac{1}{T} \sum_{i=1}^{T} x_i y_i - \overline{x}\,\overline{y} \right) / (\sigma_X \sigma_Y).$$

Again, the critical values are obtained from bootstrapping as outlined above.

4. SIZE AND POWER PROPERTIES: A SIMULATION STUDY

Though being easily implemented, the use of the LM-type test from the last section mainly depends on its size and power properties (compared to its natural competitors). For this purpose, a simulation study was conducted based on 5000 repetitions of bivariate data sets with length $N = 200, 500, 1000$. The corresponding critical values were obtained on 200 bootstrap replications. Beside the testing procedures of Box (1949), Jennrich (1970) and Goetzmann et al. (2005, briefly GLR henceforth) the following tables also contain the simulation results for the three LM-type tests. Simulating from a bivariate normal distribution with constant correlation $\rho_t \equiv 0.5$, all tests, except for the Box test, have correct empirical size close to $\alpha = 0.05$ (see table 1).

	Box	Jennrich	GLR	$\mathcal{LM}_{ave}$	$\mathcal{LM}_{sup}$	$\mathcal{LM}_{exp}$
$N = 200$	0.0730	0.0474	0.0532	0.0554	0.0572	0.0572
$N = 500$	0.0844	0.0500	0.0556	0.0570	0.0654	0.0626
$N = 1000$	0.0816	0.0514	0.0474	0.0564	0.0560	0.0556

Table 1: Nominal size: Constant correlation and normal distribution.

Secondly, in order to verify the robustness against the assumption of normality, random pairs are repeatedly drawn from a bivariate Student-t distribution with 5 degrees of freedom. The frequencies of rejection are summarized by table 2, below.

	Box	Jennrich	GLR	$\mathcal{LM}_{ave}$	$\mathcal{LM}_{sup}$	$\mathcal{LM}_{exp}$
$N = 200$	0.2210	0.1664	0.0516	0.2314	0.2302	0.2292
$N = 500$	0.2348	0.1794	0.0498	0.1972	0.2070	0.2014
$N = 1000$	0.2380	0.1844	0.0514	0.1720	0.1694	0.1718

Table 2: Nominal size: Constant correlation and Student-t distribution.

As being expected, only the GLR test preserves the error rate of type I. However, the LM-type tests seem to be somewhat more robust than the Jennrich test, at least for large samples. Moreover, replacing the bivariate normal distribution by the Student-t distribution may lead to a more robust version of the LM-type test.
In order to compare the power of the tests towards different alternatives, different scenarios were considered:

- Scenario **A**: $\rho_t = 0.5$ for $t = 1, \dots, N/2$ and $\rho_t = 0.7$ for $t = N/2 + 1, \dots, N$,
- Scenario **B**: $\rho_t = 0.5$ for $t = 1, \dots, N/4$ and $\rho_t = 0.7$ for $t = N/4 + 1, \dots, N$,
- Scenario **C**: $\rho_t = 0.5 + 0.1 \sin(2\pi t/T) + 0.1 \cos(2\pi t/T)$ for $t = 1, \dots, N$,
- Scenario **D**: $\rho_t = 0.5 + 0.1 \sin(2\pi t/T) + 0.1 \cos(\pi t/T)$ for $t = 1, \dots, N$.

The results in table 3 confirm the power the LM-type tests if the correlation of the data behaves wavelike (i.e. scenario C and D). But also if the correlation is piecewise constant (Scenario A) we observe that the power is marginally worse than that of the Jennrich test for large sample sizes N. If the correlation is constant but has two periods of different lengths (Scenario B) the LM-type tests even outperform their competitors for different sample sizes.

	Box	Jennrich	GLR	$\mathcal{LM}_{ave}$	$\mathcal{LM}_{sup}$	$\mathcal{LM}_{exp}$
Scenario A						
$N = 200$	0.7118	0.6052	0.5950	0.2502	0.3942	0.3874
$N = 500$	0.9672	0.9394	0.9420	0.6616	0.8514	0.8430
$N = 1000$	1.0000	0.9996	0.9996	0.9604	0.9964	0.9964
Scenario B						
$N = 200$	0.3354	0.2270	0.2208	0.3002	0.2832	0.2844
$N = 500$	0.6202	0.4908	0.4772	0.6644	0.6560	0.6498
$N = 1000$	0.8636	0.7740	0.7696	0.9454	0.9478	0.9444
Scenario C						
$N = 200$	0.2894	0.2208	0.2088	0.2582	0.3032	0.3028
$N = 500$	0.5540	0.4692	0.4674	0.6300	0.7354	0.7338
$N = 1000$	0.8126	0.7568	0.7578	0.9368	0.9694	0.9692
Scenario D						
$N = 200$	0.1470	0.1012	0.1038	0.1356	0.1364	0.1368
$N = 500$	0.2312	0.1698	0.1686	0.2972	0.3426	0.3316
$N = 1000$	0.3692	0.2960	0.2936	0.5756	0.6412	0.6354

Table 3: Frequencies of rejection.

5. EMPIRICAL APPLICATION: FOREIGN EXCHANGE RATES

To demonstrate the usefulness of the LM-type tests, we focus on the daily noon spot US dollar exchange rates (USD/local currency) for the British Pound (GBP) and the Swiss Franc (CHF) over the period 1 January 1990 to 31 December 2005 ($N = 4044$ observations) which are available from the PACIFIC Exchange Rate Service[1]. In a first step, the exchange rates S_t are transformed to percentual log-returns defined as $R_{i,t} \equiv \ln(S_t/S_{t-1}) \cdot 100$. Both prices and log-returns can be seen in figure 1, below.

[1]Download under the URL-link `http://pacific.commerce.ubc.ca`. All exchange rates are given in volume notation, where values are expressed in units of the target currency per unit of the base currency.

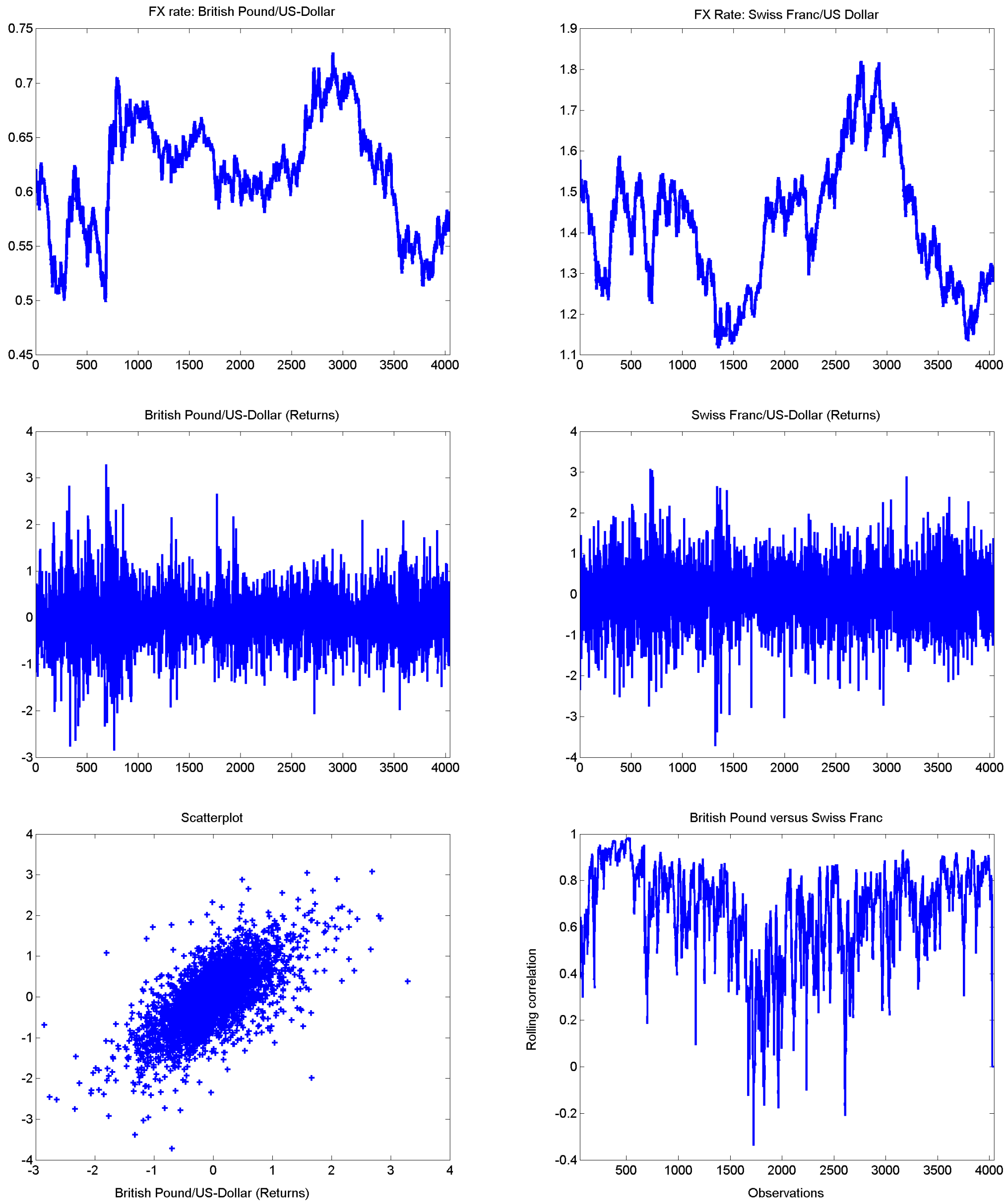

Figure 1: Exchange Rates.

Moreover, the scatterplot above reveals the positive correlation between the returns of the exchange rates series. In addition, a rolling correlation plot (which depicts the correlation of the last 20 days over the whole period) delivers first insight into the time-structure of the correlation. With this in mind and noting that $\rho_1 = 0.6762$ and $\rho_2 = 0.6652$ for the first and the second half of the sample, respectively, the results of the above-mentioned tests from table 4 are not surprising. Obviously, the "traditional" tests fail in detecting the time-varying structure of the underlying data, whereas the LM-type tests doesn't fail. In order to get rid off the heavy tails and the usual clustering (i.e., roughly speaking, periods with low volatility followed by periods with high volatility) we fitted a standard GARCH(1,1) model to the univariate time series and considered the residuals of the GARCH model, instead.

The corresponding series are labeled by an asterisk in table 4. The results, however, remain essentially unchanged.

Series	Box	Jennrich	GLR	$\mathcal{LM}_{ave}$	$\mathcal{LM}_{sup}$	$\mathcal{LM}_{exp}$
GBP - CHF	1.01 [3.84]	0.69 [3.84]	0.38 [3.84]	**32.63** [5.04]	**71.63** [9.57]	**69.01** [7.20]
GBP* - CHF*	0.08 [3.84]	0.05 [3.84]	0.04 [3.84]	**82.32** [4.27]	**131.86** [8.27]	**129.03** [6.51]

Table 4: Empirical results for the exchange rate data.

6. SUMMARY

A new LM-type test for constant correlation based on certain trigonometric functions is introduced. In particular, we discuss three different test statistics. The corresponding critical values are obtained applying non-parametric bootstrap methods. A small simulation study shows that the new test guarantees correct empirical size and is powerful against many alternatives. Finally, we successfully applied our test to foreign exchange rate data over the period of 15 years.

BIBLIOGRAPHY

Becker, R., Enders, W., Hurn, S. (2004). A General Test for Time Dependence in Parameters. *Journal of Applied Econometrics* 19:899-906.

Box, G. E. P. (1949). A General Distribution Theory for a Class of Likelihood Criteria. *Biometrika* 36(3/4):317-346.

Browne, M. W., Shapiro, A. (1986). The Asymptotic Covariance Matrix of Sample Correlation Coefficients under General Conditions. *Linear Algebra and its Applications* 82:589-599.

Mari, D. D., Kotz, S. (2001). *Correlation and Dependence.* Imperial College Press, Singapore.

Fisher, R. A. (1915). Frequency distribution of the values of the correlation coefficient in samples from an infinitely large population. *Biometrika* 10:507-521.

Goetzmann, W. N., Li, L., Rouwenhorst, K. G. (2005). Long-Term Global Market Correlations. *Journal of Business* 78(1):1-38.

Jennrich, R. I. (1970). An asymptotic chi-square test for the equality of two correlation matrices. *JASA* 65:904-912.

Kullback, S. (1967). On testing correlation matrices. *Applied Statistics* 16:80-85.

Neudecker, H., Wesselman, A. M. (1990). The Asymptotic Variance Matrix of the Sample Correlation Matrix. *Linear Algebra and its Applications* 127:589-599.

Tang, G. Y. N. (1995). Intertemporal Stability in International Stock Markets Relationships: A Revisit. *The Quarterly Review of Economics and Finance* 35:579-593.