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Constructing Symmetric Generalized FGM Copulas by means of certain Univariate Distributions

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Abstract In this paper we focus on symmetric generalized Fairlie-Gumbel-Morgenstern (or symmetric Sarmanov) copulas which are characterized by means of so-called generator functions. In particular, we introduce a class of generator functions which is based on univariate distributions with certain properties. Some of the generator functions from the literature are recovered. Moreover two new generators are suggested, implying two new copulas. Finally, the opposite way around, it is exemplarily shown how to calculate the univariate distribution which belongs to a given copula generator function.

1 Introduction

Analyzing the dependence between the components $X_1, \ldots, X_n$ of a random vector X is subject to various lines of statistical research. For this purpose, copula functions have been introduced by Sklar (1959) in order to allow for a separation between the marginal distributions and the dependence structure. Although having no closed form and admitting only symmetric dependence patterns, the Gaussian and the Student-t copula – which belong to the class of elliptical copulas – have become very popular. Alternatively, several classes of copulas like the Archimedean class have been discussed in the literature. Within this work we focus on the construction of symmetric generalizations of FGM (so-called symmetric Sarmonov) copulas which can be simply estimated and which are given in closed form.

The paper is organized as follows: In Section 2 we briefly review the notion of FGM copulas and some of their generalizations. In particular, the main results of Amblard and Girard (2002) are reviewed who discuss generalized FGM copulas depending on specific symmetric generator functions. Section 3 introduces generator functions derived from certain symmetric and asymmetric distributions. Several well-known generator functions will be recovered. Moreover, we introduce two new copula generators based on the Cauchy distribution and on the generalized secant hyperbolic distribution of Vaughan (2002). Finally, in Section 4 we exemplarily demonstrate the opposite way around, i.e. how to derive the distribution function from a given copula generator.

2 The FGM copula family and generalizations

Whereas copulas can be defined in a multivariate setting, we restrict ourselves to the bivariate case. Loosely speaking, a 2-copula is a two-dimensional distribution function defined on the unit square with uniformly distributed marginals. More formally, a two-dimensional copula is a function $C : [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

1. C is 2-increasing, i.e. for $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$ holds:

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0.$$

2. For all $u, v \in [0,1]$: $C(u,0) = C(0,v) = 0$ and $C(u,1) = C(1,v) = u$.

Note that every copula is bounded below by $C^{min}(u,y) = \max\{u + v - 2, 0\}$ and above by $C^{max}(u,v) = \min\{u,v\}$, the so-called Fréchet-Hoeffding bounds. Moreover, the copula associated with the joint distribution of two independent uniform variables is given by $C^{\perp}(u,v) = uv$. Morgenstern (1956), Gumbel (1960) and Farlie (1960) are connected to the so-called FGM-copula or family B10 in Joe (2001) which is of the form

$$C_\theta(u,v) = uv\,(1 + \theta(1-u)(1-v))\,, \qquad -1 \le \theta \le 1$$

and can be seen as a direct generalization of $C^{\perp}$. More general, $(1-u)$ and $(1-v)$ can be replaced by suitable functions ("kernels" or "copula generators" or "generator function") $A(u)$ and $B(v)$ that are assumed to be differentiable functions on $(0,1)$ with $A(u) \to 0, B(v) \to 0$ for $u, v \to 1$, i.e.

$$C_\theta(u,v) \equiv uv \cdot \Big(1 + \theta A(u)B(v)\Big) \equiv uv + \theta\varphi(u)\,\psi(v) \qquad (1)$$

with $\varphi(u) \equiv u\,A(u)$ and $\psi(u) \equiv u\,B(u)$. Note that the admissible range of θ depends on the functions A, B (see, for example, Bairamov und Kotz, 2002). Distributions of that form were discussed by Sarmanov (1966) and Lee (1996), whereas Shubina and Lee (2004) derived bounds for the corresponding dependence measures.

Within this work we focus on the symmetric case, i.e. $A(u) = B(v)$ or $\psi(u) = \varphi(v)$ for $u = v$. That means, equation (1) reduces to

$$C_\theta(u,v) = uv\Big(1 + \theta A(u)A(v)\Big) = uv + \theta\varphi(u)\,\varphi(v), \quad \theta \in [L, U]. \qquad (2)$$

Huang and Kotz (1999), for example, discuss the symmetric family generated by $A(u) = (1-u^p)$, $p > 0$ with $-\max\{1,p\}^{-2} \le \theta \le 1/p$, and the family generated by $A(u) = (1-u)^p$, $p \ge 1$ with $-1 \le \theta \le ((p+1)(p-1))^{p-1}$. Both models are nested in the Bairamov-Kotz FGM family with $A(u) =$

$(1-u^p)^q$, $p > 0, q \geq 1$. The admissible range of θ is calculated by Bairamov and Kotz (2002). Lai and Xie (2000) consider $\varphi(u) = u^b(1-u)^p$, $b, p \geq 1$. Restricting the parameter θ in equation (2) to the interval $[-1, 1]$, Amblard and Girard (2002) show that

$$C_{\theta,\varphi}(u, v) = uv + \theta\varphi(u)\varphi(v), \quad \theta \in [-1, 1] \tag{3}$$

is a copula if and only if the generator function φ satisfies the following conditions:

(C1) Boundary condition: $\varphi(0) = \varphi(1) = 0$,

(C2) Lipschitz condition: $|\varphi(u) - \varphi(v)| \leq |u - v| \quad \forall (u, v) \in [0, 1]^2$.

Furthermore, C_θ is absolutely continuous. Equivalently, $C_\theta(u, v)$ from (3) generates a parametric family of copulas if, and only, if

(C3) $|\varphi'(u)| \leq 1$ for almost every $u \in [0, 1]$,

(C4) $|\varphi(u)| \leq \min\{u, 1-u\} \quad \forall u \in [0, 1]$,

(C5) φ is absolutely continuous.

The generator function φ can be interpreted as a "measure of distance" on the main diagonal of the unit square between the copula C_φ and the independence copula $C^\perp$:

$$\varphi(u) = \sqrt{\frac{1}{\theta}(C_\varphi(u, u) - C^\perp(u, u))} = \sqrt{\frac{1}{\theta}(C_\varphi(u, u) - u^2)}.$$

Let $\mathcal{G}$ denote the class of copula generators $C_\theta(u, v)$ which satisfy the conditions (C3)-(C5). Amblard and Girard (2002) state the following members.

Example 1 (Copula generators)

1. The generator $\varphi_1(u) = u(1-u)$ is connected to the standard FGM family.
2. The generator $\varphi_2(u) = \frac{1}{\pi}\sin(\pi\, u)$ leads to a family with "large dependence".
3. The upper bound itself $\varphi_3(u) = \min\{u, 1-u\}$ is a generator function.
4. The generator of the Huang-Kotz family is given by $\varphi_4(u) = u(1-u)^\gamma, \quad \gamma \geq 1$.

More general, we next focus on the class of so-called "distribution-based generators" and show that the generator functions $\varphi_1, \ldots, \varphi_4$ can be recovered within this class.

3 Distribution-based generators

3.1 Generators derived from symmetric distributions

Let $\mathcal{G}_0 \subseteq \mathcal{G}$ denote the (sub)class of symmetric, positive generators, i.e. any $\varphi \in \mathcal{G}_0$ satisfies

(S) $\varphi(1-u) = \varphi(u) \quad \forall u \in [0, 0.5]$,

(P) $\varphi(u) \geq 0 \quad \forall u \in [0, 1].$,

Consequently, for any $\varphi \in \mathcal{G}_0$

(C3) $\iff \quad |\varphi'(u)| \leq 1, \qquad$ for almost every $u \in [0, 0.5]$,

(C4) $\iff \quad 0 \leq \varphi(u) \leq u, \quad \forall u \in [0, 0.5]$.

A subset of $\mathcal{G}_0$ which is derived from symmetric random variables X is the class of ξ_f-functions which are introduced in the next definition.

Definition 1 (ξ_f-function) *Let X denote a continuous, symmetric random variable with corresponding density function $f(x)$, distribution function $F(x)$ and inverse distribution function $F^{-1}(x)$. Assume further that f is differentiable. Define*

$$\xi_f(u) \equiv f(F^{-1}(u)), \quad u \in [0,1] \tag{4}$$

and let Ξ denote the class of functions from that type.

It is straightforward to verify that, if f is symmetric, ξ_f is monotone increasing on $[0, 0.5]$ and satisfies **(S)** and **(P)**. In order to ensure that ξ_f is actually a copula generator we have to impose some certain restrictions on X which can be expressed using generalized absolute score functions.

Definition 2 (Generalized absolute score function) *For a positive, differentiable function $g : \mathbb{R} \to \mathbb{R}_+$ we define the generalized absolute score function by*

$$\psi_g(x) = \left| -\frac{d}{dx} \ln(g(x)) \right| = \left| \frac{g'(x)}{g(x)} \right| . \tag{5}$$

We now characterize ξ_f-functions that actually generate a copula by means of generalized absolute score functions.

Lemma 1 *Let $\xi \in \Xi$. Then,*

$$\textbf{(C3)} \quad \Longleftrightarrow \quad \psi_f(x) \leq 1, \qquad \forall\, x \leq 0,$$

$$\textbf{(C4)} \quad \Longleftrightarrow \quad \psi_F(x) \leq 1, \qquad \forall\, x \leq 0.$$

Proof: Firstly,

$$|\varphi'(u)| \leq 1 \quad \forall\, u \in [0,1] \Longleftrightarrow \left|\frac{d}{du} f(F^{-1}(u))\right| \leq 1 \quad \forall\, u \in [0, 0.5]$$

$$\Longleftrightarrow \psi_f(x) = \left|\frac{f'(x)}{f(x)}\right| \leq 1 \quad \forall\, x \in (-\infty, 0].$$

Secondly,

$$|\varphi(u)| \leq \min\{u, 1-u\} \quad \forall\, u \in [0,1] \Longleftrightarrow |f(F^{-1}(u))| \leq u \quad \forall\, u \in (0, 0.5]$$

$$\Longleftrightarrow f(x) \leq F(x) \quad \forall\, x \in (-\infty, 0] \Longleftrightarrow \psi_F(x) = \frac{f(x)}{F(x)} \leq 1 \quad \forall\, x \in (-\infty, 0]. \quad \Box$$

To sum up, every symmetric and absolute continuous univariate distribution with restricted ψ_f−function and restricted ψ_F−function can be used to construct a copula generator φ_f. The interpretation of the density-based generator is the following: Loosely speaking, the density represents the distribution of the distance between a copula and the independence copula. In the sequel, we investigate for several densities with closed form cdf which meet the requirements of theorem 1. However, a counterexample is given first.

Example 2 (Normal generator) For the standard normal distribution

$$\psi_f^{Norm} = |x| > 1 \text{ for } |x| > 1.$$

Hence, equation **(C3)** is not satisfied and ξ_{Norm} is not a copula generator.

Example 3 (Logistic generator) The pdf, cdf and quantile functions of the logistic distribution are given by

$$f_{Log}(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \; F_{Log}(x) = \frac{1}{1+e^{-x}} \quad \text{and } F_{Log}^{-1}(x) = \log\left(\frac{x}{1-x}\right).$$

Hence, $\xi_{Log}(u) = f_{Log}(F_{Log}^{-1}(u)) = u(1-u)$, i.e. we recover the generator of the FGM family from example 1(1). Note that

$$\psi_F^{Log}(x) = \frac{\exp(-x)}{1+\exp(-x)} \leq 1 \quad \text{and} \quad \psi_f^{Log}(x) = \left|\frac{\exp(-x)-1}{1+\exp(-x)}\right| \leq 1.$$

Example 4 (Hyperbolic secant generator) For the hyperbolic secant distribution,

$$f_{HS}(x) = \frac{2/\pi}{(e^x + e^{-x})},\ F_{HS}(x) = \frac{\arctan(e^x)}{\pi/2} \text{ and } F_{HS}^{-1}(x) = \log\left(\tan\left(\frac{\pi x}{2}\right)\right).$$

Consequently, $\xi_{HS}(u) = \frac{1}{\pi}\sin(\pi u)$, i.e. we recover the copula generator from example 1(2). In this case,

$$\psi_F^{HS}(x) = \frac{(\exp(x)+\exp(-x))^{-1}}{\arctan(e^x)} \leq 1 \quad \text{and} \quad \psi_f^{HS}(x) = \left|\frac{e^{-x}-e^x}{e^x+e^{-x}}\right| \leq 1.$$

Example 5 (Laplace generator) The Laplace or double exponential distribution can be characterized by

$$f_{Lap}(x) = \begin{cases} 0.5e^x, & x \leq 0 \\ 0.5e^{-x}, & x > 0 \end{cases},\quad F_{Lap}(x) = \begin{cases} 0.5e^x, & x \leq 0 \\ 1-0.5e^{-x}, & x > 0 \end{cases},$$

$$\text{and its inverse } F_{Lap}^{-1}(x) = \begin{cases} \log(2x), & x \leq 0 \\ \log\left(\frac{1}{2-2x}\right), & x > 0 \end{cases}.$$

Consequently, $\xi_{Lap}(u) = \min\{u, 1-u\}$, i.e. we recover the "upper bound" from example 1(3). In addition, for $x \leq 0$

$$\psi_F^{Lap}(x) = 1 \leq 1 \quad \text{and} \quad \psi_f^{Lap}(x) = |1| \leq 1.$$

Finally, we derive two new generators corresponding to the Cauchy distribution and to the GSH distribution from Vaughan (2002). Note that the GSH distribution nests both the logistic and the hyperbolic secant distribution.

Example 6 (Cauchy generator) For the Cauchy distribution,

$$f_C(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \; F_C(x) = \frac{2\arctan(x)+\pi}{2\pi}, \text{ and } F_C^{-1}(x) = \tan(\pi(u-0.5)).$$

Hence,

$$\xi_C(u) = \frac{1}{\pi}\frac{1}{1+(\tan(\pi(u-0.5)))^2}, \tag{6}$$

which we will call a "Cauchy generator". It can be verified that

$$\psi_F^C(x) = \frac{2}{(1+x^2)(2\arctan(x)+\pi)} \leq 1 \quad \text{and} \quad \psi_f^C(x) = \left|\frac{-2x}{1+x^2}\right| \leq 1.$$

Example 7 (GSH generator) The generalized secant hyperbolic (GSH) distribution of Vaughan (2002) is defined by

$$f_{GSH}(x;t) = \begin{cases} \frac{\sin(t)}{2t(\cosh(x)+\cos(t))} & \text{for } -\pi < t < 0, \\ \frac{e^{-x}}{(1+e^{-x})^2} & \text{for } t = 0, \\ \frac{\sinh(t)}{2t(\cosh(x)+\cosh(t))} & \text{for } t > 0. \end{cases}$$

It includes the logistic ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases. Note that t is a kurtosis parameter. Moreover, both the cumulative distribution function and the inverse distribution function are given in closed form by

$$F_{GSH}(x) = \begin{cases} 1 + \frac{1}{t}\operatorname{arccot}\left(-\frac{\exp(x)+\cos(t)}{\sin(t)}\right) & \text{for } -\pi < t < 0, \\ \frac{1}{1+e^{-x}} & \text{for } t = 0, \\ 1 - \frac{1}{t}\operatorname{arccoth}\left(\frac{\exp(x)+\cosh(t)}{\sinh(t)}\right) & \text{for } t > 0 \end{cases}$$

and

$$F_{GSH}^{-1}(u) = \begin{cases} \ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \text{for } -\pi < t < 0, \\ \ln\left(\frac{u}{1-u}\right) & \text{for } t = 0, \\ \ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \text{for } t > 0. \end{cases}$$

It is straightforward but cumbersome to derive the $\xi_{GSH}-$function which is given by

$$\xi_{GSH}(u) = \begin{cases} \frac{\sin(t)}{t\left(\frac{\sin(tu)}{\sin(t(1-u))}+\frac{\sin(t(1-u))}{\sin(tu)}+2\cos(t)\right)} & \text{for } -\pi < t < 0, \\ u(1-u) & \text{for } t = 0, \\ \frac{\sinh(t)}{t\left(\frac{\sinh(tu)}{\sinh(t(1-u))}+\frac{\sinh(t(1-u))}{\sinh(tu)}+2\cosh(t)\right)} & \text{for } t > 0. \end{cases} \quad (7)$$

It is shown in Appendix A that the function ξ_{GSH} from equation (7) generates a copula for $t \in [-\pi/2, \infty)$. Figure 1, below, displays ξ_f, ψ_f, ψ_F for the distributions discussed before.

Figure 1 to be inserted here

In general, the conditional mean function (as one possible dependence function) is given by

$$E(U|V=v) = \int_0^1 uc(u,v)du = \frac{1}{2} + \theta\varphi'(v)\int_0^1 u\varphi'(u)du.$$

Using partial integration rules und setting $K_\varphi \equiv \int_0^1 \varphi(u)du$ we get

$$E(U|V=v) = \frac{1}{2} - \theta K_\varphi \varphi'(v),$$

i.e. the conditional mean expectation is determined only by $-\varphi'(v)$ (i.e. by the score function). Hence copulas based on the logistic, the hyperbolic secant, the Laplace and the GSH distribution imply a stictly monotone increasing relation (in mean), whereas the copula generated by the Cauchy distribution allows for non-monotone relations (see figure 2, below).

Figure 2 to be inserted here

3.2 Generators derived from asymmetric distributions

We now drop the assumption that X is a random variable with symmetric distribution. Again f_X and F_X denote the density and the cumulative distribution function, respectively. The copula generator from example 3 can be generalized to the asymmetric case, other examples are thinkable.

Example 8 (Asymmetric logistic generator) The generalized logistic distribution of type I has density function which is given by

$$f_{AL}(x;\alpha) = \frac{\alpha \exp(-x)}{(1+\exp(-x))^{\alpha+1}}, \quad \alpha > 0.$$

Note that α determines the skewness of the distribution. It is negatively skewed for $0 < \alpha < 1$, positively skewed for $\alpha > 1$ and symmetric for $\alpha = 1$. It can be easily verified that $F_{AL}(x;\alpha) = (1+\exp(-x))^{-\alpha}$ and $F_{AL}^{-1}(u;\alpha) = -\ln(u^{-1/\alpha} - 1)$. Consequently,

$$\xi_{AL}(u;\alpha) = \alpha \left(u^{-1/\alpha} - 1\right) u^{1+1/\alpha} = \alpha(1 - u^{1/\alpha})u.$$

Obviously, setting $\alpha = 1$ results in the copula generator from example 3. Furthermore, for $0 < \alpha \leq 1$ we have

$$\alpha(1 - u^{1/\alpha})u \leq \min(u, 1-u), \quad u \in [0,1].$$

For $u = 0$ and $u = 1$ the inequality is trivial. For $u \leq 0.5$,

$$\alpha(1 - u^{1/\alpha})u \leq (1 - u^{1/\alpha})u \leq u, \quad \alpha < 0 \leq 1.$$

For $u > 0.5$, we have to show that

$$\xi_{AL}(u;\alpha) = \alpha(1 - u^{1/\alpha})u \leq (1-u)u \leq 1-u.$$

The second inequality is trivial, whereas the first one follows from $\alpha(1 - u^{1/\alpha}) \leq (1-u)$ for $0 < \alpha \leq 1$. Note that the last inequality can be proved using $\xi_{AL}(u;\alpha) \leq 0.5$, $\xi_{AL}(1;\alpha) = 0 = 1-u$, $0 \geq \xi'_{AL}(u;\alpha) \geq -1$ and the continuity from $\xi_{AL}(\cdot;\alpha)$. Additionally,

$$\xi'_{AL}(u;\alpha) = \alpha\left(1 - u^{1/\alpha}\right) - u^{1/\alpha}.$$

For $0 < \alpha \leq 1$ and $0 \leq u \leq 1$, the absolute value of ξ'_{Al} is restricted to 1, because $\alpha\left(1-u^{1/\alpha}\right)$ is positive and less than 1 and $u^{1/\alpha}$ is also positive and less than one for $u \in [0,1]$. To sum up, ξ_{AL} is a copula generator for $0 < \alpha \leq 1$, which is similar to that of the symmetric modified FGM copula. Different copula generators are plotted in the figure 3, below.

Figure 3 to be inserted here

4 Distributions derived from generators

Up to now, we derived the copula generator for a given distribution which satisfies the requirements of Corollary 1. Conversely, given a copula generating function, what is the corresponding distribution? The answer to this question will be exemplarily demonstrated for so-called polynomial copulas (see Mari and Kotz, 2001). In general, a polynomial copula of order (p,q) is defined by

$$C_{pq}(u,v) = uv\left(1 + \frac{\theta_{pq}}{(p+1)(q+1)}(1-u^p)(1-v^q)\right), \qquad (8)$$

with $0 < \theta_{kq} \leq \min\left(\frac{(k+1)(q+1)}{q}, \frac{(k+1)(q+1)}{k}\right)$. In the symmetric case, i.e. setting $p = q \equiv k$) in (8), the polynomial copula is a symmetric generalized

FGM copula: Comparing (8) and (2), we find

$$\phi(u) = \frac{1}{k+1} \cdot u\left(1 - u^k\right).$$

According to (4), the corresponding distribution can be derived as the solution of the differential equation

$$F'(F^{-1}(u)) = \phi(u) \iff F'(x) = \phi(F(x)) \text{ for } u \equiv F(x)$$

with respect to F. In the case of a symmetric polynomial copula, this differential equation can be transformed to a Bernoulli differential equation (see, for example, Bronstein and Semendjajew, 1998) which can be solved explicitly. The solution is given by

$$F(x) = k\left(1 + Ce^{-k/(k+1)x}\right)^{-k},$$

where C is a constant that ensures F to be actually a cumulative distribution function.

Proof: Setting $y \equiv F(x)$ and $y' \equiv f(x)$, we have to solve

$$y' - \frac{1}{k+1}y = -\frac{1}{k+1}y^{k+1} \quad \iff \quad \frac{y'}{y^{k+1}} - \frac{1}{k+1}y^{-k} = -\frac{1}{k+1}.$$

Substituting $z \equiv y^{-k}$, we obtain $z' = -ky^{-(k+1)}y'$ and

$$-\frac{1}{k}z' - \frac{1}{k+1}z = -\frac{1}{k+1}.$$

Multiplying both sides with $-k$, a special case of the simple first order differential equation

$$z' + P(x)z = Q(x), \text{ with } P(x) = Q(x) = k/(k+1)$$

results. The corresponding solution (see Bronstein & Semendjajew, 1991) is given by

$$
\begin{aligned}
z &= e^{-\int P(x)dx} \left(\int Q(x) e^{\int P(x)dx} dx + C \right) \\
&= e^{-k/(k+1)x} \left(\frac{k}{k+1} \int e^{k/(k+1)x} dx + C \right) = 1 + Ce^{-k/(k+1)x}.
\end{aligned}
$$

Undoing the substitution,

$$
y = \left(1 + Ce^{-k/(k+1)x} \right)^{-k}. \quad \square
$$

5 Summary

Symmetric generalized Fairlie-Gumbel-Morgenstern (FGM) copulas can be characterized by means of so-called generator functions or generators. In this paper we introduced generators based on specific univariate distributions. These distributions are characterized, for example, by a limited score function, which rules out the normal distribution as a suitable candidate. Several generators are recovered within this setting. Moreover, we derived two new generator functions from the Cauchy distribution and the GSH distribution of Vaughan (2002), implying two new symmetric generalized FGM copulas. Finally, the opposite way around it is exemplarily shown, i.e. how to calculate the univariate distribution which belongs to a given copula generator function.

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A Proof for GSH generator

Theorem 1 (GSH copula generator) *The function* ξ_{GSH} *from equation (7) generates a copula for* $t \in [-\pi/2, \infty)$.

Proof: For $t = -\pi/2$ and $t = 0$ the result was already proved. To verify for what values of t this function generates a copula, we have to check the conditions of theorem 1. Note that

$$\psi_f^{GSH}(x;t) = \left|\frac{-\sinh(x)}{\cosh(x)+\cos(t)}\right| \text{for } \ t \in (-\pi, 0]$$

First, $\cosh(t) \geq 1$ for all $t > 0$. Moreover, $-\sinh(x) \geq 0$ and $\cosh(x) \geq 1$ for $x \leq 0$ and hence

$$\left|\frac{-\sinh(x)}{\cosh(x)+\cosh(t)}\right| = \frac{-\sinh(x)}{\cosh(x)+\cosh(t)} \leq \frac{-\sinh(x)}{\cosh(x)} = -\tanh(x) \leq 1.$$

A similar inequality is valid because $\cos(t) \geq 0$ for all $t \in [-\pi/2, 0]$. For $t < -\pi/2$, it is no longer valid.

Case 1: $-\pi/2 < t < 0$. Next it remains to show that

$$\psi_F^{GSH}(x;t) = \frac{\frac{\sin(t)}{2t(\cosh(x)+\cos(t))}}{1+\frac{1}{t}\operatorname{arccot}\left(-\frac{\exp(x)+\cos(t)}{\sin(t)}\right)} \leq 1 \text{ for } t \in (-\pi/2, 0), x \in (-\infty, 0].$$

Substituting $u \equiv e^x \Longleftrightarrow x \equiv \ln(u)$, we have to show that

$$\psi_F^{GSH}(u;t) = \frac{\frac{\sin(t)}{2t\left(\frac{u^2+1}{2u}+\cos(t)\right)}}{1+\frac{1}{t}\operatorname{arccot}\left(-\frac{u+\cos(t)}{\sin(t)}\right)} \leq 1 \text{ for any } t \in (-\pi/2, 0), u \in (0, 1].$$

To prove this, we show in the lemma below that $\lim_{u \to 0+} \psi_F^{GSH}(u;t) = 1$ and ψ_F^{GSH} is decreasing in u for all $t \in (\pi/2, 0)$. Noting that $\psi_F^{GSH}(u;t)$ is continuous on $[0, 1]$, the proof is completed.

Lemma 2 *Define* $f(u;t) \equiv \frac{\sin(t)}{2t\left(\frac{u^2+1}{2u}+\cos(t)\right)}$ *and* $g(u;t) \equiv 1+\frac{1}{t}\operatorname{arccot}\left(-\frac{u+\cos(t)}{\sin(t)}\right)$. *Then*

$$\lim_{u \to 0+} \psi_F^{GSH}(u;t) = \lim_{u \to 0+} \frac{f(u;t)}{g(u;t)} = 1. \tag{9}$$

$$\frac{d}{u}\psi_F^{GSH}(u;t) \leq 0, \qquad 0 < u \leq 1. \tag{10}$$

Proof: Assume that $t \in (-\pi/2, 0)$ is arbitrary but fix. Again, $\sin(t) < 0$ and $\cos(t) > 0$ for $t \in (-\pi/2, 0)$. As $\lim_{u \to 0+} f(u;t) = 0$ and $\lim_{u \to 0+} g(u;t) = 0$, we can apply the rule of l'Hopital

$$\lim_{u \to 0+} \psi_F^{GSH}(u;t) = \lim_{u \to 0+} \frac{f(u;t)}{g(u;t)} = \lim_{u \to 0+} \frac{f'(u;t)}{g'(u;t)}.$$

On the one hand,

$$\begin{aligned}
\lim_{u \to 0+} f'(u;t) &= \lim_{u \to 0+} -\frac{1}{2}\frac{\sin(t)}{t}\frac{\left(1-\frac{u^2+1}{2u^2}\right)}{\left(\frac{u^2+1}{2u}+\cos(t)\right)^2} \\
&= \frac{\sin(t)}{t}\lim_{u \to 0+} -\frac{1}{2}\frac{\left(u^2-\frac{u^2+1}{2}\right)}{\left(\frac{u^2+1}{2}+u\cos(t)\right)^2} = \frac{\sin(t)}{t}.
\end{aligned}$$

On the other hand,

$$\lim_{u\to 0+} g'(u;t) = \lim_{u\to 0+} \frac{1}{t\sin(t)\left(1+\frac{(u+\cos(t))^2}{(\sin(t))^2}\right)} = \frac{\sin(t)}{t}.$$

Thus, equation (9) is verified. Secondly, we have to show that

$$\frac{d\psi_F(u)}{du} = \frac{f'(u;t)g(u;t) - f(u;t)g'(u;t)}{g(u;t)^2} \le 0,$$

or, equivalently, that

$$\frac{\sin(t)(1-u^2)}{t(u^2+1+2u\cos(t))^2} \cdot \left(1 + \frac{\operatorname{arccot}\left(-\frac{u+\cos(t)}{\sin(t)}\right)}{t}\right) - \left(\frac{u\sin(t)^2}{t^2\left(u^2+1+2u\cos(t)\right)^2}\right) \le 0 \iff$$

$$\frac{\sin(t)(1-u^2)}{t^2} \cdot \left(t + \pi/2 + \arctan\left(\frac{u+\cos(t)}{\sin(t)}\right)\right) - \frac{u\sin(t)^2}{t^2} \le 0 \iff$$

$$(1-u^2)\cdot\left(t+\pi/2+\arctan\left(\frac{u+\cos(t)}{\sin(t)}\right)\right) \ge u\sin(t) \qquad (11)$$

for $u \in (0,1], t \in (-\pi/2, 0)$. Define $A(u,t) \equiv \left(t + \pi/2 + \arctan\left(\frac{u+\cos(t)}{\sin(t)}\right)\right)$ and $U(u,t) = u\sin(t)$. From

$$A(0,t) = t+\pi/2+\arctan(-\tan(\pi/2+t)) = t+\pi/2-\arctan(\tan(\pi/2+t)) = 0$$

and

$$A'(u,t) = \frac{\sin(t)}{1+u^2+2u\cos(t)} \le 0$$

we conclude that $A(u,t) \le 0$ for all $t \in (\pi/2, 0)$. Consequently, for $u \in (0,1]$, $(1-u^2) \le 1$ and $(1-u^2)A(u,t) \ge A(u,t)$. Finally, $U(0,t) = 0$ and $U'(u,t) = \sin(t) \ge \frac{\sin(t)}{1+u^2+2u\cos(t)} = A'(u,t)$. Together with the continuity of A and U, $A(u,t) \ge U(u,t)$ and the proof of the lemma is completed.

Case 2: $t > 0$. Now it has to be proved that

$$\psi_F^{GSH}(x;t) = \frac{\frac{\sinh(t)}{2t(\cosh(x)+\cosh(t))}}{1 - \frac{1}{t}\,\text{arccoth}\left(\frac{\exp(x)+\cosh(t)}{\sinh(t)}\right)} \leq 1 \text{ for } t > 0, x \in (-\infty, 0].$$

or, substituting $u = \exp(x)$ again,

$$\psi_F^{GSH}(u;t) = \frac{\frac{\sinh(t)}{2t\left(\frac{u^2+1}{2u}+\cosh(t)\right)}}{1 - \frac{1}{t}\,\text{arccoth}\left(\frac{u+\cosh(t)}{\sinh(t)}\right)} \leq 1 \text{ for } t > 0, u \in (0, 1].$$

Defining $f_1(u,t) \equiv \frac{\sinh(t)}{2t\left(\frac{u^2+1}{2u}+\cosh(t)\right)}$ and $g(u,t) \equiv 1 - \frac{1}{t}\,\text{arccoth}\left(\frac{u+\cosh(t)}{\sinh(t)}\right)$,

$$\psi_F^{GSH}(0;t) = \lim_{u\to 0+} \frac{f_1(u,t)}{g_1(u,t)} = \lim_{u\to 0+} \frac{f_1'(u,t)}{g_1'(u,t)} = \lim_{u\to 0+} \frac{\frac{\sinh(t)\left(1-u^2\right)}{t(u^2+2\,u\cosh(t)+1)^2}}{\frac{\sinh(t)}{t(u^2+2\,u\cosh(t)+1)}} = 1.$$

Remains to verify that

$$\frac{d\psi_F(u)}{du} = \frac{f_1'(u;t)g_1(u;t) - f_1(u;t)g_1'(u;t)}{g_1(u;t)^2} \leq 0,$$

or,

$$f_1'(u;t)g_1(u;t) - f_1(u;t)g_1'(u;t) \leq 0$$

This is equivalently to

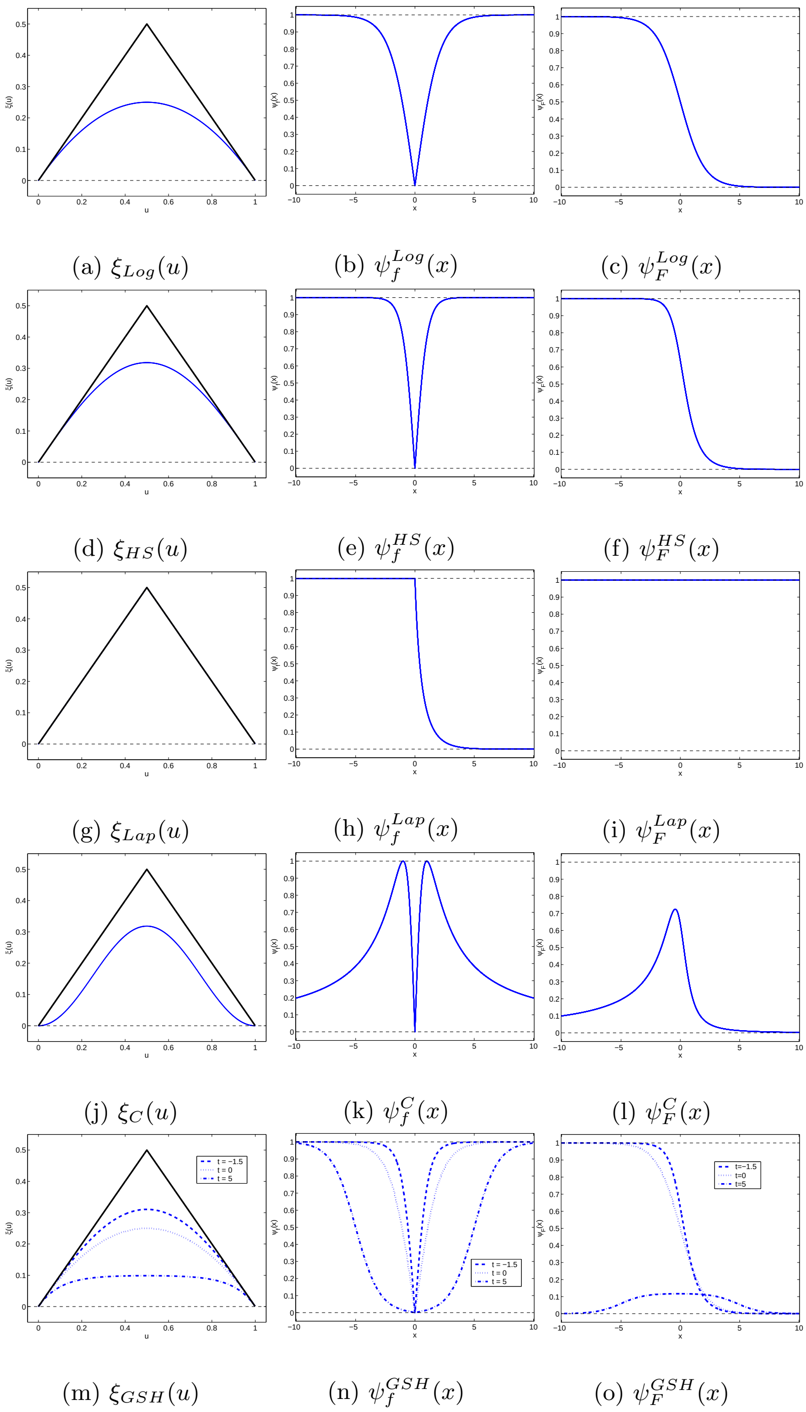
$$\left(1 - u^2\right)\left(t - \text{ arccoth}\left(\frac{u+\cosh(t)}{\sinh(t)}\right)\right) \leq u\sinh(t)$$

Define $A(u,t) \equiv \left(t - \text{ arccoth}\left(\frac{u+\cosh(t)}{\sinh(t)}\right)\right)$ and $U(u,t) = u\sinh(u)$. Then

$$(1 - u^2)A(u,t) \leq A(u,t) \leq U(u,t).$$

The second inequality follows from $A(0,t) = U(0,t)$ and

$$U'(u,t) = \sinh(t) \geq \frac{\sinh(t)}{u^2 + 2\,u\cosh(t) + 1} = A'(u,t).\Box$$



(a) $\xi_{Log}(u)$ (b) $\psi_f^{Log}(x)$ (c) $\psi_F^{Log}(x)$

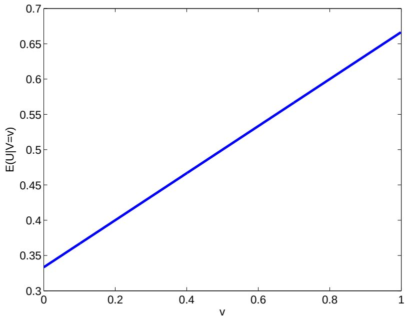
(d) $\xi_{HS}(u)$ (e) $\psi_f^{HS}(x)$ (f) $\psi_F^{HS}(x)$

(g) $\xi_{Lap}(u)$ (h) $\psi_f^{Lap}(x)$ (i) $\psi_F^{Lap}(x)$

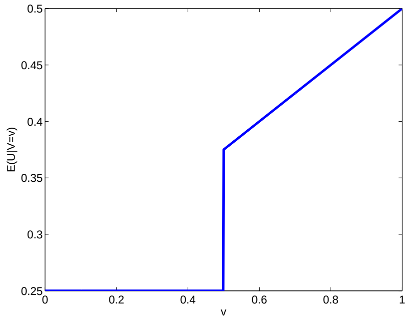
(j) $\xi_{C}(u)$ (k) $\psi_f^{C}(x)$ (l) $\psi_F^{C}(x)$

(m) $\xi_{GSH}(u)$ (n) $\psi_f^{GSH}(x)$ (o) $\psi_F^{GSH}(x)$

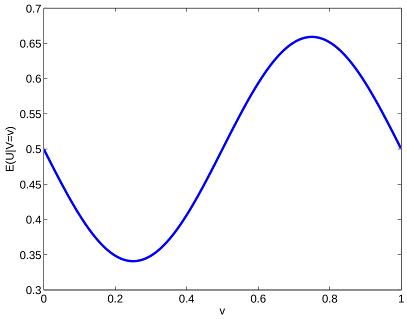
Fig. 1 Copula generator



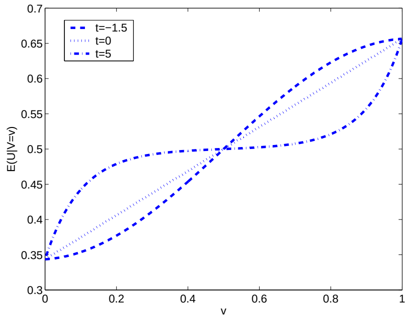
(a) Logistic



(b) Laplace



(c) Cauchy



(d) GSH

Fig. 2 Conditional Mean Functions

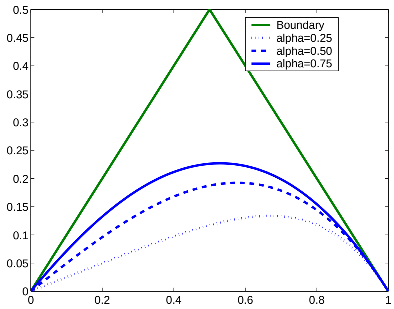


Fig. 3 Asymmetric logistic generator