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Ingo Klein and Matthias Fischer

Lehrstuhl für Statistik und Ökonometrie
Lehrstuhl für Statistik und empirische Wirtschaftsforschung
Lange Gasse 20 · D-90403 Nürnberg

SKEWNESS BY SPLITTING THE SCALE PARAMETER

Ingo Klein and Matthias Fischer
Department of Statistics and Econometrics
University of Erlangen-Nürnberg
90419 Nürnberg, Germany
Matthias.Fischer@wiso.uni-erlangen.de

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ABSTRACT

There are several possibilities to introduce skewness into a symmetric distribution. One of these procedures applies two different parameters of scale – with possibly different weights – to the positive and the negative part of a symmetric density. Within this work we show that this technique incorporates a well-defined parameter of skewness, i.e. that the generated distributions are skewed to the right (left) if the parameter of skewness takes values less (greater) than one. Secondly, we prove that the skewness parameter is compatible with the skewness ordering of van Zwet (1964) which is the strongest ordering in the hierarchy of orderings discussed by Oja (1981). Hence, the generated (skewed) distributions can be ordered by the skewness parameter.

1. INTRODUCTION

Several techniques can be applied to symmetric distributions in order to generate asymmetric ones. Tukey (1960), for example, exploits the technique of variable transformation and suggests the so-called g-transformations. Similarly, Morgenthaler and Tukey (2000) advocate kurtosis transformations with different transformation parameters on the positive and the negative axis. Azzalini (1985, 1986), on the contrary, introduces skew densities by means of $g(x) = 2f(x)F(\lambda x)$, where f and F denote the density and the distribution function, respectively, of an arbitrary symmetric distribution and $\lambda \in \mathbb{R}$ governs the amount of skewness. A further generalization in terms of weighting functions is given by Ferreira and Steel (2004).

The method we focus on can also be embedded in the framework of Ferreira and Steel (2004). The main idea is to apply different scale parameters to the positive and the negative part of a symmetric density. However, the new density distributes half of the probability mass to the negative axis and half of the mass to the positive axis. This disadvantage can be removed if the "split of the scale parameter" is appropriately weighted, as it was done by Fernández et al. (1995) and Theodossiou (1998). None of them, however, shows that the corresponding parameter is actually a skewness parameter.

For that reason, the proceeding is as follows. Section 2 reviews the technique of splitting the scale parameter. In section 3, we specify our notion of skewness and prove that the transformed distributions are skewed to the right if the corresponding parameter takes values less than one. Section 4 introduces the ϕ-function of a distribution and derives general conditions – based on the $\phi-$function – how two distributions can be ordered according to the skewness ordering of van Zwet (1964). In section 5, the proof is given for the method of splitting the scale parameter.

2. SPLITTING THE SCALE PARAMETER

Assume that X is a symmetric random variable with corresponding density f. A new density can be defined by

$$f_a(x;\gamma) \equiv a(\gamma) \cdot \frac{1}{\gamma} \cdot f(x/\gamma) \cdot \mathrm{I}_{(-\infty,0]}(x) + (2 - a(\gamma)) \cdot \gamma \cdot f(x\gamma) \cdot \mathrm{I}_{(0,\infty)}(x) \tag{1}$$

with $0 \leq a(\gamma) \leq 2$ for $\gamma > 0$ and $a(1) = 1$. Note that, in principle, two different parameters of scale are introduced for the negative and the positive part of the distribution. For that reason we call this method "splitting the scale parameter". For $\gamma = 1$, no transformation takes place. In the following, we assume $a(\gamma)$ to be either strictly increasing or constant equal to one for $\gamma > 0$.

Example 1 *1. For $a \equiv 1$ we obtain a 'simple' split, given by*

$$f_a(x;\gamma) \equiv \frac{1}{\gamma} \cdot f(x/\gamma) \cdot \mathrm{I}_{(-\infty,0]}(x) + \gamma \cdot f(x\gamma) \cdot \mathrm{I}_{(0,\infty)}(x).$$

Obviously, $F_a(0;\gamma) = \int_{-\infty}^{0} f_a(x;\gamma)dx = 1/2$, *independent of* γ.

2. Choosing $a(\gamma) = \frac{2\gamma^2}{\gamma^2+1}$ *results in*

$$f_a(x;\gamma) = \frac{2}{\gamma + \frac{1}{\gamma}} \cdot \Big[f(x/\gamma) \cdot \mathrm{I}_{(-\infty,0]}(x) + f(x\gamma) \cdot \mathrm{I}_{[0,\infty)}(x) \Big]. \qquad (2)$$

Densities of the form (2) were considered by Fernández et al. (1995) to generate skew exponential power distributions. Grottke (2001) applied this transformation to the GT-distribution, whereas Fischer and Vaughan (2002) applied it to the GSH distribution. Notice that

$$a'(\gamma) = \frac{4\gamma}{1+\gamma^2} > 0, \quad \gamma > 0.$$

We next show that the function $a(.)$ is completely determined if the density $f_a(x;\gamma)$ should be continuous at $x = 0$.

Lemma 1 *Assume F to be a distribution function on $\mathbb{R}$ with continuous density. The density f_a from (1) is continuous on $\mathbb{R}$ if and only if $a(\gamma) = 2\gamma^2/(1+\gamma^2)$.*

Proof: The result follows immediately from

$$\lim_{x \to 0^-} f_a(x;\gamma) = \frac{a(\gamma)}{\gamma} f(0) = (2 - a(\gamma))\gamma f(0) = \lim_{x \to 0^+} f_a(x;\gamma) \quad \square$$

For that reason, we focus on the two cases of example 1, above. The corresponding distribution function of f_a from equation (1) is given by

$$F_a(x;\gamma) = a(\gamma) \cdot F(x/\gamma) \cdot \mathrm{I}_{(-\infty,0]}(x) + \Big[a(\gamma) - 1 + (2 - a(\gamma)) \cdot F(x\gamma) \Big] \cdot \mathrm{I}_{(0,\infty)}(x). \qquad (3)$$

Occasionally, we make use of the inverse distribution function which is

$$F_a^{-1}(u;\gamma) = \gamma \cdot F^{-1}\left(\frac{u}{a(\gamma)} \right) \cdot \mathrm{I}_A(u) + \frac{1}{\gamma} \cdot F^{-1}\left(\frac{1 + u - a(\gamma)}{2 - a(\gamma)} \right) \cdot \mathrm{I}_{\overline{A}}(u) \qquad (4)$$

with $A = (0, \frac{\gamma^2}{1+\gamma^2}) = (0, \frac{a(\gamma)}{2})$ and $\mathrm{I}_{\overline{A}}(u) = 1 - \mathrm{I}_A(u)$.

3. SPLITTING THE SCALE PARAMETER AND SKEWNESS TO THE RIGHT

To the best of our knowledge, all authors using the method speak of the "skewness parameter γ" without having it defined. In the next definition we specify our notion of skewness to the right in terms of the distribution function F_a from equation (3), above.

Definition 1 *The distribution function F_a with median $F_a^{-1}(0.5)$ will be called skewed to the right if*

$$F_a(F_a^{-1}(0.5;\gamma)+c;\gamma) \leq 1 - F_a(F_a^{-1}(0.5;\gamma)-c;\gamma)$$

for all $c \in \mathbb{R}$ with $'' < ''$ for at least one $c \in \mathbb{R}$.

We next show that the distribution function F_a is skewed to the right, if $\gamma < 1$ and $a(\gamma) \equiv 1$. In addition, the same result will be proved for $a(\gamma) = 2\gamma^2/(1+\gamma^2)$ and a unimodal symmetric density f with median at the $x = 0$. Note that there are several definitions of unimodality. According to Hàjek and Šidak (1967, p. 15), a density f is unimodal, if $-\log f$ is increasing. This definition cancels out, for instance, the Student t-distribution. For that reason, we call a density f *unimodal* if the corresponding distribution function F is strictly convex for $x < 0$ and strictly concave for $x > 0$.

Theorem 1 *1. Let $a(\gamma) \equiv 1$ and F denote a strictly increasing distribution function with symmetric density f. Then F_a from equation (3) is skewed to the right if $\gamma < 1$.*

2. Let $a(\gamma) = \frac{2\gamma^2}{1+\gamma^2}$ and F be a continuous distribution function with unimodal symmetric density. Then F_a is skewed to the right if $\gamma < 1$.

Proof: 1. The median of F is $F^{-1}(0.5) = 0$ and

$$F_a(x;\gamma) = F(x/\gamma;\gamma)I_{(-\infty,0]}(x) + F(x\gamma)I_{(0,\infty)}(x).$$

Let $c > 0$ be fixed. Then, $F_a(F^{-1}(0.5)+c;\gamma) = F(c\gamma)$ and due to the symmetry of F

$$1 - F_a(F^{-1}(0.5)-c;\gamma) = 1 - F_a(-c) = 1 - F(-c/\gamma) = F(c/\gamma).$$

For $0 < \gamma < 1$, we have $|c\gamma| < |c/\gamma|$. Using the strict monotonicity of F,

$$F_a(c;\gamma) = F(c\gamma) < F(c/\gamma) = 1 - F_a(-c;\gamma) \quad \text{for } c > 0.$$

For $c < 0$, $F_a(-c;\gamma) < 1 - F_a(c;\gamma)$ and $F_a(c;\gamma) < 1 - F_a(c;\gamma)$. The case $c = 0$ is trivial.

2. Let $x_{0.5} = F_a^{-1}(0.5;\gamma)$ denote the median of F_a.

Case 1: $x_{0.5} + c \geq 0$ and $x_{0.5} - c \geq 0$ for $c > 0$.

From the unimodality of f we conclude that f_a has to be unimodal with modus $x_{0.5}$. Therefore, F_a is strictly concave for $x > 0$. This means that

$$\lambda \cdot F_a(x_1;\gamma) + (1-\lambda) \cdot F_a(x_2;\gamma) \leq F_a(\lambda x_1 + (1-\lambda)x_2);\gamma)$$

for $x_1, x_2 > 0, 0 \leq \lambda \leq 1$. Setting $\lambda \equiv \frac{1}{2}$, $x_1 \equiv x_{0.5} - c$ and $x_2 \equiv x_{0.5} + c$ we get

$$\frac{F_a(x_{0.5} - c;\gamma)}{2} + \frac{F_a(x_{0.5} + c;\gamma)}{2} \leq F_a\left(\frac{x_{0.5} - c}{2} + \frac{x_{0.5} + c}{2};\gamma\right) = F_a\left(x_{0.5};\gamma\right) = \frac{1}{2}.$$

Multiplying with 2, $F_a(x_{0.5} - c;\gamma) + F_a(x_{0.5} + c;\gamma) \leq 1$. If F is strictly concave this inequality holds strictly for at least one $c > 0$.

Case 2: $x_{0.5} - c \leq 0$ and $x_{0.5} + c \geq 0$ for $c > 0$.

Define $b(c) \equiv F_a(x_{0.5} - c;\gamma) + F_a(x_{0.5} + c;\gamma)$ for $c > 0$. Maximization of $b(c)$ with respect to c implies the necessary condition

$$b'(c) = \frac{a(\gamma)}{\gamma} \cdot f\left(\frac{x_{0.5} - c}{\gamma}\right) + (2 - a(\gamma))\gamma \cdot f((x_{0.5} + c)\gamma) \stackrel{!}{=} 0, \tag{5}$$

or, equivalently, $f((x_{0.5} - c)/\gamma) \stackrel{!}{=} f((x_{0.5} + c)\gamma)$. Due to the symmetry of f, this condition can only be satisfied if the absolute values of the arguments are identical. For $a(\gamma) = \frac{2\gamma^2}{1+\gamma^2}$ this leads to the solution

$$c_0 = c_0(\gamma) = x_{0.5} \cdot \frac{1 + \gamma^2}{1 - \gamma^2}.$$

c_0 is strictly positive if $\gamma < 1$. It can be verified that the second derivative of b at c_0 is strictly negative for $\gamma < 1$. Therefore, we have a maximum at c_0 for $\gamma < 1$. It remains to show that $b(c_0) \leq 1$. Plugging c_0 into (5) we get

$$\begin{aligned} b(c_0) &= a(\gamma) \cdot F((x_{0.5} - c_0)/\gamma) + a(\gamma) - 1 + (2 - a(\gamma)) \cdot F((x_{0.5} + c_0)\gamma) \\ &= a(\gamma)F(-2x_{0.5}\gamma/(1-\gamma^2)) + a(\gamma) - 1 + (2 - a(\gamma))F(2x_{0.5}\gamma/(1-\gamma^2)). \end{aligned}$$

Using again the symmetry,

$$F(2x_{0.5}\gamma/(1-\gamma^2)) = 1 - F(-2x_{0.5}\gamma/(1-\gamma^2)) \leq 1.$$

Hence, $b(c_0) \leq 2a(\gamma) - 1 + 2 - 2a(\gamma) = 1.\square$.

4. ψ- AND ϕ-FUNCTION OF A DISTRIBUTION

Let F denote the cumulative distribution function of a random variable X and assume that F is continuous on $\mathbb{R}$ and has a density f which itself is differentiable on $\mathbb{R} \setminus \{0\}$. The *score function* of X is defined by

$$\psi_F(x) \equiv \begin{cases} -\frac{f'(x)}{f(x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

It is anti-symmetric for every density which is symmetric (around 0), i.e.

$$\psi_F(x) = -\psi_F(-x), \quad x \in \mathbb{R}.$$

Example 2 (Gaussian and Student-t distribution) *The score function of a zero-mean and unit-scale Student-t distribution with k degrees of freedom is given by*

$$\psi_t(x) = \frac{2k+1}{k} \cdot \frac{x}{1 + x^2/k} = \frac{(k+1)\,x}{k + x^2}, \quad x \in \mathbb{R}. \tag{6}$$

It is not strictly monotone increasing on $\mathbb{R}$ because $\lim \psi_t(x) = 0$ for $x \to \infty$ and $k \in \mathbb{N}$ fix. Letting $k \to \infty$ in (6) we obtain the score function of a standard Gaussian variable,

$$\psi_\Phi(x) = x, \quad x \in \mathbb{R}.$$

Example 3 (GT distribution) *The generalized Student-t distribution of McDonald and Newey (1988) with parameters $p, q > 0$ has density*

$$f_{GT}(x;p,q) = \frac{p}{2q^{1/p}B(1/p,q)}\left(1+\frac{|x|^p}{q}\right)^{-(q+1/p)}$$

and reduces to the Student-t distribution with $\nu = 2q$ degrees of freedom for $p = 2$. Due to the symmetry, we focus on the positive part of the distribution. The corresponding ψ-function is given by

$$\psi_{GT}(x;p,q) = \frac{(qp+1)\,x^{p-1}}{q+x^p}, \quad x > 0.$$

Example 4 (GSH distribution) *The generalized secant hyperbolic (GSH) distribution of Vaughan (2002) generalizes both the logistic and the hyperbolic secant distribution. Its density is given by*

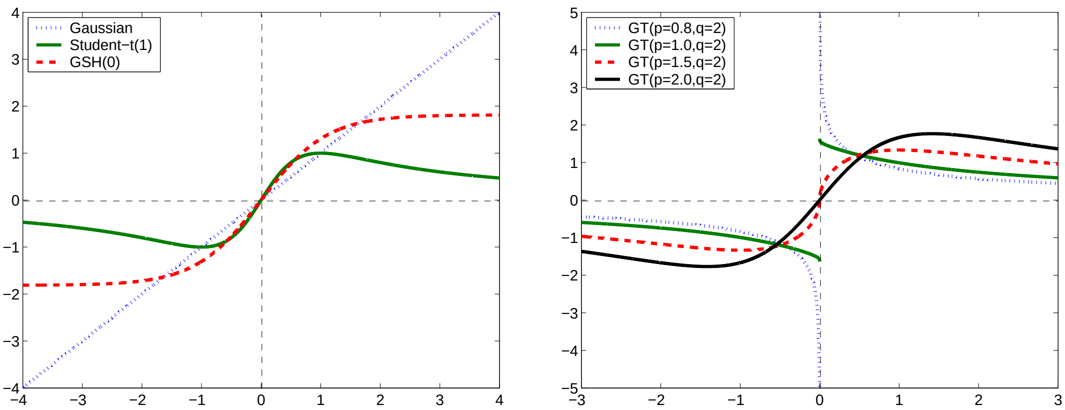
$$f_{GSH}(x;t) = c_1(t)\cdot\frac{\exp(c_2(t)x)}{\exp(2c_2(t)x)+2a(t)\exp(c_2(t)x)+1}, \qquad x \in \mathbb{R}$$

with normalizing constants depending on the kurtosis parameter t through

$$\begin{array}{llll} a(t) = \cos(t), & c_2(t) = \sqrt{\frac{\pi^2-t^2}{3}} & c_1(t) = \frac{\sin(t)}{t}\cdot c_2(t), & \textit{for } -\pi < t \le 0, \\ a(t) = \cosh(t), & c_2(t) = \sqrt{\frac{\pi^2+t^2}{3}} & c_1(t) = \frac{\sinh(t)}{t}\cdot c_2(t), & \textit{for } t > 0 \end{array}.$$

It can be verified that the score function is given by

$$\psi_{GSH}(x) = \frac{c_2(t)\,\left(\exp(2\,c_2(t)\,x)-1\right)}{\exp(2\,c_2(t)\,x)+2\,a(t)\exp(c_2(t)\,x)+1}.$$



(a) Gaussian, Student-t and GSH (b) GT distribution

Figure 1: ψ-functions for different distributions

In the next lemma we derive necessary and sufficient conditions for differentiable score functions such that the ratio between score- and density function,

$$\phi_F(x) \equiv \frac{\psi_F(x)}{f(x)}, \quad x \in \mathbb{R} \tag{7}$$

is strictly monotone increasing. This ratio will be called the *ϕ-function* in the sequel.

Lemma 2 *Let F be a cumulative distribution function with density f which is assumed to be twice differentiable on $\mathbb{R}$. Then $\phi_F(x)$ is strictly monotone increasing if and only if*

$$\psi_F'(x) > -\psi_F(x)^2 \quad x \in \mathbb{R}. \tag{8}$$

Proof: Applying the quotient rule for $x \in \mathbb{R}$,

$$\phi_F'(x) = \frac{\psi_F'(x)f(x) - \psi_F(x)f'(x)}{f(x)^2} = \frac{1}{f(x)}\Big(\psi_F'(x) + \psi_F(x)^2\Big).$$

The term in brackets is positive if and only if $\psi_F'(x) > -\psi_F(x)^2$. □

Note that if the score function ψ_F itself is strictly monotone increasing, condition (8) is always satisfied. Hence, ϕ_F is strictly monotone increasing, too. This is true for the Gaussian distribution and the GSH distribution. For the Student-t-distribution, however, the validity of inequation (8) has to be shown.

Example 5 (Student-t distribution, continued) *It is straightforward to verify that the first derivative of the Student-$t(k)$ score function is given by*

$$\psi_{t(k)}'(x) = \frac{k+1}{k} \cdot \frac{1 - x^2/k}{(1 + x^2/k)^2} \quad x \in \mathbb{R}.$$

Consequently,

$$\psi_{t(k)}'(x) + \psi_{t(k)}(x)^2 = \frac{k(k+1)(1+x^2)}{(k+x^2)^2} > 0$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$.

Example 6 (Laplace distribution) *The density of the Laplace distribution is given by*

$$f_{LAPLACE}(x) = \frac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}$$

with corresponding score function $\psi_{LAPLACE}(x) = sign(x), \quad x \in \mathbb{R}$. *The latter is discontinuous at* $x = 0$. *This point of discontinuity transmits to the* ϕ*-function*

$$\phi_{LAPLACE}(x) = 2sign(x)e^{|x|}, \quad x \in \mathbb{R}.$$

However,

$$\phi'_{LAPLACE}(0) \equiv \lim_{x\to 0^-} \phi'_{LAPLACE}(x) = \lim_{x\to 0^+} \phi'_{LAPLACE}(x) = 2.$$

Therefore, with $\phi_{LAPLACE}(0) = 0$,

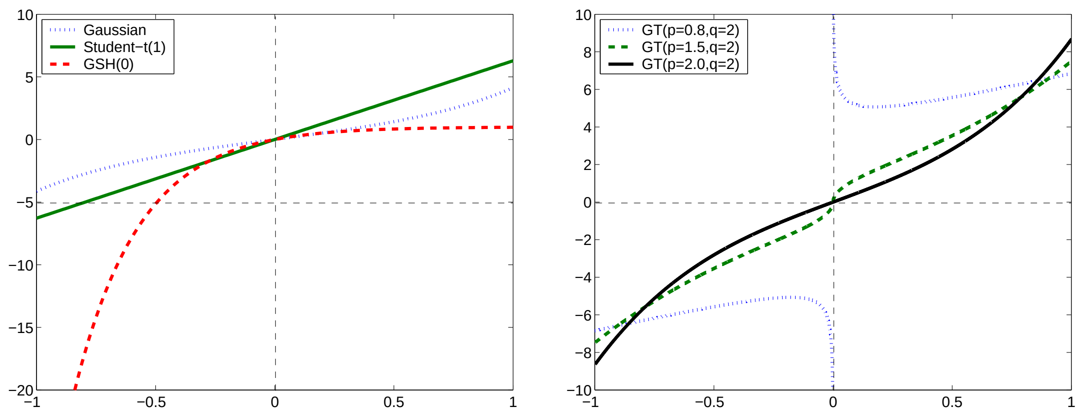
$$\phi'_{LAPLACE}(x) = 2e^{|x|} > -\phi_{LAPLACE}(x)^2 = -4e^{2|x|} \quad x \in \mathbb{R}$$

and the $\phi-$*function of a Laplace distribution is strictly monotone increasing.*

Example 7 (GT distribution, continued) *For the GT distribution,*

$$\psi'_{GT}(x) + \psi_{GT}(x)^2 = \frac{(qp+1)\,x^{p-2}\,(p-1)}{qp+1+x^p} - \frac{(qp+1)\,(x^{p-1})^2\,p}{(qp+1+x^p)^2} + \frac{(qp+1)^2\,(x^{p-1})^2}{(q+x^p)^2}.$$

This expression becomes negative for $p = 0.75$, $q = 10$ *and* $x = 0.2$, *for example. Consequently, the* ϕ*-function of the GT distribution is not strictly monotone increasing on* $\mathbb{R}$.



(a) Gaussian, Student-t and GSH (b) GT distribution

Figure 2: ϕ-functions for different distributions

By means of the ϕ-function we are able to verify whether two distributions can be ordered according to the skewness ordering of van Zwet (1964): In the notion of Van Zwet (1964), a continuous distribution F is less skewed to the right than a continuous distribution G (briefly, $F \preceq_c G$), if

$$G^{-1}(F(x)) \quad \text{is convex on } \mathbb{R}. \tag{9}$$

If the densities f and g of F and G, respectively, exist and are differentiable, the second derivative of $G^{-1}(F(x))$ is positive if and only if

$$\frac{f(x)^2}{g(x)}\left(\frac{f'(x)}{f(x)^2} - \frac{g'(G^{-1}(F(x))}{g(G^{-1}(F(x))^2}\right) > 0$$

for $x \in \mathbb{R}$. In terms of the ϕ-function, the convexity of $G^{-1}(F(x))$ requires that

$$\textbf{(A1)} \qquad \phi_G(G^{-1}(F(x))) - \phi_F(x) > 0 \quad \text{for } x \in \mathbb{R}.$$

Setting $u \equiv F^{-1}(x)$, condition (A1) can be rewritten as

$$\textbf{(A2)} \qquad \phi_G(G^{-1}(u)) - \phi_F(F^{-1}(u)) > 0 \quad \text{for } u \in (0,1).$$

Note that this condition can only be verified for parametric functions ϕ_F and ϕ_G, respectively.

5. SPLITTING THE SCALE PARAMETER AND SKEWNESS ORDERING

We have already shown in section 3 that, under certain conditions, F_a from equation (3) defines a family of skew distributions. With the help of section 4, we are now able to prove that this family of skew distributions can be ordered by means of the skewness parameter γ if the partial ordering of van Zwet (1964) is considered. According to (9), this ordering concerns

$$\Lambda(x;\gamma_1,\gamma_2) \equiv F_a^{-1}(F_a(x;\gamma_1);\gamma_2) \quad \text{for } x \in \mathbb{R}$$

which has to be either convex or concave on $\mathbb{R}$ for $\gamma_2 < \gamma_1$. Notice that

$$\Lambda(x;\gamma_1,\gamma_2) = \begin{cases} \gamma_2 F^{-1}\left(\frac{F(x/\gamma_1)a(\gamma_1)}{a(\gamma_2)}\right) & \text{for } x \le \gamma_1 F^{-1}\left(\frac{a(\gamma_2)}{2\gamma_1)}\right), \\ \frac{1}{\gamma_2}F^{-1}\left(\frac{1+a(\gamma_1)F(x/\gamma_1)-a(\gamma_2)}{2-a(\gamma_2)}\right) & \text{for } \gamma_1 F^{-1}\left(\frac{a(\gamma_2)}{2\gamma_1)}\right) < x < 0, \\ \frac{1}{\gamma_2}F^{-1}\left(\frac{a(\gamma_1)+(2-a(\gamma_1))\cdot F(x\gamma_1)-a(\gamma_2)}{2-a(\gamma_2)}\right) & \text{for } x \ge 0. \end{cases} \tag{10}$$

In theorem 2 we show that $1/\gamma$ from (1) is a skewness parameter in the sense of van Zwet's ordering if either the ϕ-function of F is strictly monotone increasing and $a'(\gamma) > 0$ or $a(\gamma) \equiv 1$. In this case the ψ- and the ϕ-function of F are only defined for $x \neq 0$ because f is continuous at $x = 0$ only for $a(\gamma) = 2\gamma^2/(1+\gamma^2)$, but not differentiable at $x = 0$ in all cases. This requires a special treatment at $x = 0$.

Let ψ_a and ψ denote the score functions and, ϕ_a and ϕ the ϕ-functions of F_a and F, respectively. Obviously,

$$\psi_a(x;\gamma) = \frac{\psi(x/\gamma)}{\gamma} \cdot \mathrm{I}_{(-\infty,0)}(x) + \psi(x\gamma) \cdot \gamma \cdot \mathrm{I}_{(0,\infty)}(x)$$

and

$$\phi_a(x;\gamma) = \frac{\phi(x/\gamma)}{a(\gamma)} \cdot \mathrm{I}_{(-\infty,0)}(x) + \frac{\phi(x\gamma)}{2-a(\gamma)} \cdot \mathrm{I}_{(0,\infty)}(x).$$

According to (A2), a sufficient condition for

$$F_a^{-1}(F_a(x;\gamma_1);\gamma_2), \quad \gamma_2 < \gamma_1$$

to be convex both on $\{x < 0\}$ and $\{x > 0\}$ is that $\phi_a(u;\gamma)$ is a strictly decreasing function of γ both on $\{0 < u < a(\gamma_1)/2\}$ and on $\{a(\gamma_1)/2 < u < 1\}$. For a fixed $u \in (0,1)$ we have to demonstrate that

$$\frac{\partial \phi_a(F_a^{-1}(u;\gamma);\gamma)}{\partial \gamma} = \frac{\partial \phi_a(x;\gamma)}{\partial \gamma}\Big|_{x=F_a^{-1}(u;\gamma)} \frac{\partial F_a^{-1}(u;\gamma)}{\partial \gamma} < 0$$

for $u < a(\gamma)/2$ and $u > a(\gamma/2)$. If $a(\gamma)$ is strictly increasing with inverse function a^{-1} the relation has to hold for $\gamma < a^{-1}(2u)$ and $\gamma > a^{-1}(2u)$, $u \in (0,1)$.

Theorem 2 *1. Let $a(\gamma) \equiv 1$ and F strictly increasing. If $\gamma_2 < \gamma_1$,*

$$F_a^{-1}(F_a(x;\gamma_1);\gamma_2) \text{ is convex.}$$

2. Let $a(\gamma) = \frac{2\gamma^2}{1+\gamma^2}$ for $\gamma > 0$ and F be a continuous distribution function with density function f which is continuous on $\mathbb{R}$ and differentiable for $\mathbb{R} \setminus \{0\}$ such that $\phi'(x) > 0$ for $x \neq 0$. If $\gamma_2 < \gamma_1$,

$$F_a^{-1}(F_a(x;\gamma_1);\gamma_2) \text{ is convex.}$$

Proof: 1. For $a(\gamma) \equiv 1$,

$$F_a(x;\gamma_1) = F(x/\gamma_1)I_{(-\infty,0]}(x) + F(x\gamma_1)I_{(0,\infty)}(x)$$

and

$$F_a^{-1}(u;\gamma_2) = \gamma_2 F^{-1}(u)I_{(0,1/2]}(u) + 1/\gamma_2 F^{-1}(u)I_{(1/2,1)}(u).$$

It directly follows that

$$F_a^{-1}(F_a(x;\gamma_1);\gamma_2) = \begin{cases} \gamma_2 F^{-1}(F(x/\gamma_1)) = \frac{\gamma_2}{\gamma_1}x & \text{for } x \le 0 \\ 1/\gamma_2 F^{-1}(F(x\gamma_1)) = \frac{\gamma_1}{\gamma_2}x & \text{for } x \le 0 \end{cases}.$$

This function is convex if $\gamma_2/\gamma_1 < \gamma_1/\gamma_2$. This holds for $\gamma_2 < \gamma_1$.

2. *Case 1:* $x \neq 0$. From $\phi'(x) > 0$, we conclude that

$$\frac{\partial \phi_a(x;\gamma)}{\partial x} = \frac{\phi'(x/\gamma)}{\gamma a(\gamma)} \cdot \mathrm{I}_{(-\infty,0)}(x) + \frac{\phi'(x\gamma)\gamma}{2-a(\gamma)} \cdot \mathrm{I}_{(0,\infty)}(x) \; > 0.$$

The partial derivative of $\phi_a(x;\gamma)$ with respect to γ is given by

$$\begin{aligned} \frac{\partial \phi_a(x;\gamma)}{\partial \gamma} &= \left(-\frac{a'(\gamma)}{a(\gamma)^2}\phi(x/\gamma) - \frac{1}{a(\gamma)}\phi'(x/\gamma)\frac{x}{\gamma^2}\right) \cdot \mathrm{I}_{(-\infty,0)}(x) \\ &\quad + \left(\frac{a'(\gamma)}{(2-a(\gamma))^2}\phi(x\gamma) + \frac{1}{2-a(\gamma)}\phi'(x\gamma)\gamma\right) \cdot \mathrm{I}_{(0,\infty)}(x) > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{\partial F_a^{-1}(u;\gamma)}{\partial \gamma} &= \left[F^{-1}(u/a(\gamma)) + \gamma \cdot \frac{1}{f(F^{-1}(u/a(\gamma)))} \cdot \frac{-ua'(\gamma)}{a(\gamma)^2}\right] \cdot \mathrm{I}_{(0,1/2a(\gamma))}(u) \\ &\quad + \left[-\frac{F^{-1}\left(\frac{1+u-a(\gamma)}{2-a(\gamma)}\right)}{\gamma^2} - \frac{1}{\gamma} \cdot \frac{\frac{a'(\gamma)(1-u)}{(2-a(\gamma))^2}}{f\left(\left(F^{-1}\left(\frac{1+u-a(\gamma)}{2-a(\gamma)}\right)\right)\right)}\right] \cdot \mathrm{I}_{(1/2a(\gamma),1)}(u). \end{aligned}$$

For $a'(\gamma) > 0$, this derivative is negative because $1-u > 0$ and $-1/\gamma^2 < 0$. Thus,

$$\frac{\partial \phi_a(F_a^{-1}(u;\gamma);\gamma)}{\partial \gamma} = \frac{\partial \phi_a(x;\gamma)}{\partial \gamma}\Big|_{x=F_a^{-1}(u;\gamma)} \frac{\partial F_a^{-1}(u;\gamma)}{\partial \gamma} < 0$$

for $0 < u < \frac{1}{2a(\gamma)}$ or $\frac{1}{2a(\gamma)} < u < 1$.

Case 2: Up to now, the proof of the global convexity is not complete because we do not know whether $F_a^{-1}(F_a(x;\gamma_1);\gamma_2)$ is convex for $\mathbb{R}$. For this purpose, using

$$\Lambda'(x;\gamma_1,\gamma_2) = \frac{f_a(x;\gamma_1)}{f_a(F_a^{-1}(F_a(x;\gamma_1);\gamma_2);\gamma_2)}, \tag{11}$$

we show that

$$\lim_{x\to 0^-} \Lambda'(x;\gamma_1,\gamma_2) \leq \lim_{x\to 0^+} \Lambda'(x;\gamma_1,\gamma_2). \tag{12}$$

If $\gamma_2 < \gamma_1$ and $a'(\gamma) > 0$, $\gamma > 0$ we get $a(\gamma_2) < a(\gamma_1)$. With $F_a(0;\gamma_1) = a(\gamma_1)/2$ it is $F_a^{-1}(a(\gamma_1)/2;\gamma_2) > 0$. With this in mind and (11),

$$\lim_{x\to 0^-} \Lambda'(x;\gamma_1,\gamma_2) = \frac{a(\gamma_1)/\gamma_1 f(0/\gamma_1)}{(2-a(\gamma_2))\gamma_2 f(F_a^{-1}(F_a(0;\gamma_1)\cdot\gamma_2)}$$

and

$$\lim_{x\to 0^+} \Lambda'(x;\gamma_1,\gamma_2) = \frac{(2-a(\gamma_1))\gamma_1 f(0\gamma_1)}{(2-a(\gamma_2))\gamma_2 f(F_a^{-1}(F_a(0;\gamma_1)\cdot\gamma_2)}.$$

Equation (12) is valid, if

$$\frac{a(\gamma_1)}{\gamma_1} \leq (2-a(\gamma_1))\gamma_1 \quad \Longleftrightarrow \quad a(\gamma_1) \leq \frac{2\gamma_1^2}{1+\gamma_1^2}.$$

This is true for $a(\gamma) = 2\gamma^2/(1+\gamma^2)$, $\gamma > 0$. □

The conclusion is that $1/\gamma$ is a skewness parameter not only by pragmatic reasons but by a precise definition of skewness as a meaningful statistical concept.

6. SUMMARY

There are several possibilities to introduce skewness into a symmetric distribution. One of these procedures applies two different parameters of scale to the positive and the negative part of a symmetric density. We showed that this technique incorporates a well-defined parameter of skewness. It is well-defined in the sense that the transformed distributions are skewed to the right if the parameter of skewness takes values less than 1. Secondly we showed that the parameter of skewness is compatible with the ordering of van Zwet (1964) which is the strongest ordering in the hierarchy of orderings discussed by Oja (1981).

BIBLIOGRAPHY

Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scand. J. Stat.* 12:171-178.

Azzalini, A. (1986). Further results on a class of distributions which includes the normal ones. *Statistica* 46:199-208.

Balanda, K. P., MacGillivray, H. L. (1990). Kurtosis and spread. *Can. J. Stat.* 18(1):17-30.

Fernandez, C., Osiewalski, J., Steel, M. F. J. (1995): Modelling and inference with ν-spherical distributions. *J. Am. Stat. Assoc.* 90:1331-1340, 1995.

Ferreira, J. T. A. S., Steel, M. F. J. (2004): A constructive representation of univariate skewed distributions. *Working Paper*, Department of Statistics, University of Warwick.

Fischer, M., Vaughan, D. (2002). Classes of skew generalized secant hyperbolic distributions. *Working Paper No. 45*, Lehrstuhl für Statistik und Ökonometrie, Universität Erlangen-Nürnberg.

Grottke, M. (2001): *Die t-Verteilung und ihre Verallgemeinerungen als Modell für Finanzmarktdaten.* Lothmar: Josef Eul Verlag.

Hàjek, J., Šidak, V. (1967): *Theory of Rank Tests.* New York: Academic Press.

McDonald, J. B., Newey, W. K. (1988): Partially adaptive estimation of regression models via the generalized-t-distribution. *Econ. Theory* 1(4):428-457.

Morgenthaler, S., Tukey, J. W. (2000): Fitting quantiles: Doubling, HR, HQ, and HHH distributions. *J. Comput. Graph. Stat.* 9(1):180-195.

Oja, H. (1981): On location, scale, skewness and kurtosis of univariate distributions. *Scand. J. Stat.* 18:154-168.

Theodossiou, P. (1998): Financial data and the skewed generalized t distribution. *Math. Sci.* 44 (12):1650-1660.

Tukey, J. W. (1960). The practical relationship between the common transformations of counts of amounts. *Technical Report No. 36.* Princeton University Statistical Techniques Research Group, Princeton.

Van Zwet, W. R. (1964). Convex transformations of random variables. *Mathematical Centre Tracts No. 7.* Mathematical Centre, Amsterdam.

Vaughan, D. C. (2002). The generalized hyperbolic secant distribution and its application. *Commun. Stat.-Theor. M.* 31(2):219-238.