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Kurtosis ordering of the generalized secant hyperbolic distribution – A technical note

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Abstract: Two major generalizations of the hyperbolic secant distribution have been proposed in the statistical literature which both introduce an additional parameter that governs the kurtosis of the generalized distribution. The generalized hyperbolic secant (GHS) distribution was introduced by Harkness and Harkness (1968) who considered the p-th convolution of a hyperbolic secant distribution. Another generalization, the so-called generalized secant hyperbolic (GSH) distribution was recently suggested by Vaughan (2002). In contrast to the GHS distribution, the cumulative and inverse cumulative distribution function of the GSH distribution are available in closed-form expressions. We use this property to proof that the additional shape parameter of the GSH distribution is actually a kurtosis parameter in the sense of van Zwet (1964).

Keywords: kurtosis ordering; hyperbolic secant distribution.

1 Introduction

The hyperbolic secant distribution — which was studied by Baten (1934) and Talacko (1956) — has not received sufficient attention in the literature, although it has a lot of nice properties: All moments and the moment-generating function exist, it is reproductive (i.e. the class is preserved under convolution) and both the cumulative and the inverse cumulative distribution function admit a closed form. In addition, the hyperbolic secant distribution exhibits more leptokurtosis than the normal and even more than the logistic distribution. Nevertheless, generalizations have been proposed that introduce an additional parameter which increase the "kurtosis" of the hyperbolic secant distribution. At first, Baten (1934) discussed the sum of n independent random variables, each having the same hyperbolic secant distribution. More general, Harkness and Harkness (1968) investigated the p-th convolution of hyperbolic secant variables for arbitrary positive $p > 0$ which, unfortunately, doesn't admit a closed-form representation for the cumulative and the inverse cumulative distribution function. Recently, Vaughan (2002) suggested a family of symmetric distributions, the so-called generalized secant hyperbolic (GSH) distribution. This family includes both the hyperbolic secant and the logistic distribution. Moreover, it closely approximates the Student t-distribution with corresponding kurtosis. In addition, the moment-generating function and all moments exist, and the cumulative and inverse cumulative distribution are again available in closed form. The range of "kurtosis" — measured by the fourth standardized moments — varies from 1.8 to infinity. Within this note we proof that the additional parameter of the GSH distribution is indeed a kurtosis parameter which preserves the kurtosis ordering of van Zwet (1964).

2 The generalized secant hyperbolic (GSH) distribution

The above-mentioned standard *generalized secant hyperbolic* (GSH) distribution – which is able to model both thin and fat tails – was introduced by Vaughan (2002) and has density

$$f_{GSH}(x;t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t)\exp(c_2(t)x) + 1}, \qquad x \in \mathbb{R} \tag{1}$$

with

$$\begin{array}{l} a(t) = \cos(t), \quad c_2(t) = \sqrt{\frac{\pi^2 - t^2}{3}}, \; c_1(t) = \frac{\sin(t)}{t} \cdot c_2(t), \; \text{ for } -\pi < t \le 0, \\ a(t) = \cosh(t), \; c_2(t) = \sqrt{\frac{\pi^2 + t^2}{3}}, \; c_1(t) = \frac{\sinh(t)}{t} \cdot c_2(t), \text{ for } t > 0 \end{array}.$$

The density from (1) is chosen so that $X \sim f_{GSH}(x)$ has zero mean and unit variance, the range of the "kurtosis parameter" t is $\in (-\pi, \infty)$. The GSH distribution includes the logistic distribution ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as limiting case for $t \to \infty$. Vaughan (2002) derived, amongst other properties, the cumulative distribution function, depending on the parameter t, given by

$$F_{GSH}(x;t) = \begin{cases} 1 + \frac{1}{t}\,\text{arccot}\left(-\frac{\exp(c_2(t)x) + \cos(t)}{\sin(t)}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\exp(\pi x/\sqrt{3})}{1 + \exp(\pi x/\sqrt{3})} & \text{for } t = 0, \\ 1 - \frac{1}{t}\,\text{arccoth}\left(\frac{\exp(c_2(t)x) + \cosh(t)}{\sinh(t)}\right) & \text{for } t > 0. \end{cases}$$

the inverse distribution function, given by

$$F_{GSH}^{-1}(u;t) = \begin{cases} \frac{1}{c_2(t)} \ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \text{für } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi} \ln\left(\frac{u}{1-u}\right) & \text{für } t = 0, \\ \frac{1}{c_2(t)} \ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \text{für } t > 0. \end{cases}$$

However, it was not proved that the "kurtosis parameter" t is actually a kurtosis parameter in the sense of van Zwet(1964). This will be done in the next section.

3 GSH distribution and kurtosis ordering

Van Zwet (1964) introduced a partial ordering of kurtosis $\preceq_S$ on the set of symmetric distribution functions $\mathcal{F}^s$. Let $F, G \in \mathcal{F}^s$ and μ_F denote the location of symmetry of F, then $\preceq_S$ is defined by

$$\textbf{(A)} \quad F \preceq_S G :\Longleftrightarrow \; G^{-1}(F(x)) \; \text{ is convex for } x > \mu_F$$

and means that G has higher kurtosis than F. Balanda and MacGillivray (1990) generalized this partial ordering of van Zwet by using so-called spread functions defined as symmetric differences of quantiles:

$$S_F(u) = F^{-1}(u) - F^{-1}(1-u), \quad u \ge 0.5.$$

In the sense of Balanda and MacGillivray (1990), an arbitrary continuous, monotone increasing distribution function F has less kurtosis than an equally distribution function G if

$$\textbf{(B)} \quad F \preceq_S G :\iff S_G(S_F^{-1}(x)) \quad \text{is convex for } x > F^{-1}(0.5).$$

If F is symmetric, $F^{-1}(u) = -F^{-1}(1-u)$ for $u > 0.5$, so that $S_F(u) = 2F^{-1}(u) \;\; u \geq 0.5$. This means that the spread function essentially coincides with the quantile function. It can be shown that (A) and (B) coincide in this case. The aim of this note is to prove that the parameter t from (1) is a kurtosis parameter in the sense of van Zwet (1964).

Proposition 3.1 (Kurtosis ordering) *Assume that X_1 (X_2) follows a generalized secant hyperbolic distribution with parameter t_1 (t_2) and corresponding cumulative distribution functions F_1 (F_2). If $t_1 < t_2$, then $F_2 \preceq_S F_1$, i.e. F_1 has higher kurtosis than F_2 (in the sense of van Zwet).*

Proof: To prove this result, we distinguish between 4 cases:

Case 1: $-\pi \leq t_1 < t_2 < 0$,

Case 2: $-\pi \leq t_1$, $t_2 = 0$,

Case 3: $t_1 = 0$, $t_2 > 0$,

Case 4: $0 < t_1 < t_2$

and refer to transitivity which was shown by Oja (1981, Theorem 5.1). According to equation (**A**), we have to show that

$$F_1^{-1}(F_2(u)) \;\; \text{is convex for } 1/2 \leq u < 1$$

or, equivalently,

$$A(u) \equiv \frac{\partial F_1^{-1}(u)/\partial u}{\partial F_2^{-1}(u)/\partial u} \;\; \text{is strictly monotone increasing for } 1/2 \leq u < 1.$$

Case 1: Assume $-\pi \leq t_1 < t_2 < 0$.

Preliminary remarks: If $-\pi \leq t_1 < t_2 < 0$ and $1/2 \leq u < 1$, then $t_1(u-1/2) \in [-\pi/2, 0]$ and $t_2(u-1/2) \in [-\pi/2, 0]$. Moreover, both $\sin(x)$ and $\cos(x)$ are strictly monotone increasing on $[-\pi/2, 0]$.

Part 1: Paraphrasing $A(u)$. Firstly,

$$\frac{\partial F_i^{-1}(u)}{\partial u} = t_i/c_2(t_i)\Big[\cot(t_i u) + \cot(t_i(1-u))\Big] \;\; \text{for } 0 < u < 1, i = 1, 2.$$

Consequently,

$$A(u) = \frac{t_1/c_2(t_1)}{t_2/c_2(t_2)} \cdot \frac{\cot(t_1 u) + \cot(t_1(1-u))}{\cot(t_2 u) + \cot(t_2(1-u))}.$$

Using $\cot(\alpha)+\cot(\beta)=\frac{\sin(\alpha+\beta)}{\sin(\alpha)\sin(\beta)}$ (see Bronstein and Semendjajew, [2.5.2.1.1]),

$$A(u)=\frac{t_1/c_2(t_1)}{t_2/c_2(t_2)}\cdot\frac{\frac{\sin(t_1)}{\sin(t_1u)\sin(t_1(1-u))}}{\frac{\sin(t_2)}{\sin(t_2u)\sin(t_2(1-u))}}=\frac{t_1/c_2(t_1)\sin(t_1)}{t_2/c_2(t_2)\sin(t_2)}\cdot\frac{\sin(t_2u)\sin(t_2(1-u))}{\sin(t_1u)\sin(t_1(1-u))}.$$

Now, because $\pi\le t_1<t_2<0$, $\sin(t_1)<0$ and $\sin(t_2)<0$. Hence,

$$K(t_1,t_2)\equiv\frac{t_1/c_2(t_1)\sin(t_1)}{t_2/c_2(t_2)\sin(t_2)}>0.$$

Finally, using $\sin(\alpha)\sin(\beta)=1/2(\cos(\alpha-\beta)-\cos(\alpha+\beta))$ (see Bronstein and Semendjajew, [2.5.2.1.1]),

$$A(u)=K(t_1,t_2)\cdot\frac{\cos(2t_2(u-1/2))-\cos(t_2)}{\cos(2t_1(u-1/2))-\cos(t_1)}>0,$$

because $\cos(x)$ is strictly monotone increasing for $x\in[-\pi,0)$.

Part 2: Proof of the convexity. We have to show that $A(u)$ is strictly monotone increasing on $[1/2,1)$. This is true, if $A'(u)>0$ for $[1/2,1)$:

$$\begin{aligned}A'(u)=\frac{K(t_1,t_2)}{N}\Big\{&\Big(-\sin(2t_2(u-1/2))2t_2\Big)\cdot\Big(\cos(2t_1(u-1/2))-\cos(t_1)\Big)\\&-\Big(-\sin(2t_1(u-1/2))2t_1\Big)\cdot\Big(\cos(2t_2(u-1/2))-\cos(t_2)\Big)\Big\}\end{aligned}$$

with $N\equiv[\cos(2t_1(u-1/2))-\cos(t_1)]^2>0$. Using the monotony of the cosinus on $[-\pi,0)$,

$$-2t_2(\cos(2t_1(u-1/2))-\cos(t_1))>0\ \text{ and }\ -2t_1(\cos(2t_2(u-1/2))-\cos(t_2))>0$$

for $1/2\le u<1$ and $-\pi\le t_1<t_2<0$. Defining

$$K^*(t_1,t_2,u)\equiv\min\Big\{-2t_2(\cos(2t_1(u-0.5))-\cos(t_1));-2t_1(\cos(2t_2(u-0.5))-\cos(t_2))\Big\},$$

we get

$$\begin{aligned}A'(u)&\ge\frac{K(t_1,t_2)K^*(t_1,t_2,u)}{N}\Big(\sin(2t_2(u-1/2))-\sin(2t_1(u-1/2))\Big)\\&=\frac{K(t_1,t_2)K^*(t_1,t_2,u)}{N}\Big(2\sin(t_2(u-1/2))\cos(t_2(u-1/2))\\&\qquad-2\sin(t_1(u-1/2))\cos(t_1(u-1/2))\Big),\end{aligned}$$

where we used $\sin(2\alpha)=2\sin(\alpha)\cos(\alpha)$ (see Gradshteyn and Ryzhik, [1.333.1]). According to the preliminary remarks,

$$\sin(t_2(u-1/2))\cos(t_2(u-1/2))-\sin(t_1(u-1/2))\cos(t_1(u-1/2))>0$$

for $-\pi \leq t_1 < t_2 < 0$ and $1/2 \leq u < 1$ implying that $A'(u) > 0 \quad$ for $1/2 \leq u < 1$.

Case 2: Assume $t_1 \in [-\pi, 0)$ and $t_2 = 0$.

The inverse distribution function of a GSH variable with $t_2 = 0$ is given by

$$F_2^{-1}(u) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{u}{1-u}\right), \quad 0 < u < 1.$$

Consequently, for $0 < u < 1$,

$$\frac{\partial F_2^{-1}(u)}{\partial u} = \frac{\sqrt{3}}{\pi}\left(\frac{1}{u} + \frac{1}{1-u}\right) = \frac{\sqrt{3}}{\pi}\left(\frac{1}{u(1-u)}\right).$$

Thus, for $-\pi < t_2 < 0$ and $0 < u < 1$,

$$\begin{aligned} A(u) &= \frac{\partial F_1^{-1}(u)/\partial u}{\partial F_2^{-1}(u)/\partial u} = \frac{t_1/c_2(t_1)[\cot(t_1 u) + \cot(t_1(1-u))]}{\sqrt{3}/\pi \cdot 1/(u(1-u))} \\ &= \frac{t_1/c_2(t_1)}{\sqrt{3}/\pi} \frac{\sin(t_1)(u(1-u))}{\sin(t_1 u)\sin(t_1(1-u))} = \frac{\pi \sin(t_1)}{c_2(t_1) t_1 \sqrt{3}} \cdot \frac{(t_1 u)(t_1(1-u))}{\sin(t_1 u)\sin(t_1(1-u))}. \end{aligned}$$

The first derivation is given by

$$\begin{aligned} A'(u) &= K(t_1)\left(\frac{t_1 \sin(t_1 u) - (t_1)^2 u \cos(t_1 u)}{\sin(t_1 u)^2} \cdot \frac{t_1(1-u)}{\sin(t_1(1-u))}\right. \\ &\quad \left. + \frac{-t_1 \sin(t_1(1-u)) + (t_1)^2(1-u)\cos(t_1(1-u))}{\sin(t_1(1-u))^2} \cdot \frac{t_1 u}{\sin(t_1 u)}\right) \\ &= \frac{K(t_1) t_1 t_1 u t_1 (1-u)}{\sin(t_1 u)\sin(t_1(1-u))}\left(\left[\frac{1}{t_1 u} - \cot(t_1 u)\right] - \left[\frac{1}{t_1(1-u)} - \cot(t_1(1-u))\right]\right) \end{aligned}$$

for $-\pi < t_1 < 0$ and $0 < u < 1$ with

$$K(t_1) \equiv \frac{\pi \sin(t_1)}{c_2(t_1) t_1 \sqrt{3}} > 0.$$

Note that a series expansion to the $\cot(x)$ for $0 < |x| < \pi$ (see Gradshteyn and Ryzhik, [1.441.7]) is given by

$$\cot(x) = \frac{1}{x} - \sum_{i=1}^{\infty} \frac{2^{2i}|B_{2i}|}{(2i)!} x^{2i-1}, \tag{2}$$

where B_i, $i = 1, 2, \ldots$, denotes the numbers of Bernoulli. Applying (2) for $x = t_1 u$ and $x = t_1(1-u)$,

$$A'(u) = K(t_1) t_1 \frac{t_1 u t_1 (1-u)}{\sin(t_1 u)\sin(t_1(1-u))}\left(\sum_{i=1}^{\infty} \frac{2^{2i}|B_{2i}|}{(2i)!}(t_1 u)^{2i-1} - \sum_{i=1}^{\infty} \frac{2^{2i}|B_{2i}|}{(2i)!}(t_1(1-u))^{2i-1}\right).$$

For $-\pi < t_1 < 0$,

$$K(t_1) t_1 \frac{t_1 u t_1 (1-u)}{\sin(t_1 u)\sin(t_1(1-u))} < 0.$$

The term in brackets is negative because $t_1(1-u) > t_1 u$ and $(t_1(1-u))^{2i-1} > (t_1 u)^{2i-1}$, $i = 1, 2, \ldots$ for $1/2 < u < 1$ and $t_1 < 0$. Combining both results, $A'(u) > 0$ for $-\pi < t_1 < 0$ and $1/2 < u < 1$. This completes the proof of case 2.

Case 3: Assume $t_1 = 0$ and $t_2 > t_1$.

As calculated above,

$$\frac{\partial F_2^{-1}(u)}{\partial u} = \frac{t_2}{c_2(t_2)} \cdot \Big[\coth(t_2 u) + \coth(t_2(1-u)) \Big], \qquad 0 < u < 1,$$

with $c_2(t_2) = \sqrt{\frac{\pi^2 + t_2^2}{3}} > 0$ for $t_2 > 0$. It now follows that

$$\begin{aligned} A(u) &= \frac{\partial F_1^{-1}(u)/\partial u}{\partial F_2^{-1}(u)/\partial u} = \frac{\sqrt{3}/\pi}{t_2/c_2(t_2)} \cdot \frac{1/(u(1-u))}{\coth(t_2 u) + \coth(t_2(1-u))} \\ &= \frac{\sqrt{3} t_2/\pi}{\sinh(t_2)/c_2(t_2)} \cdot \frac{\sinh(t_2 u)\sinh(t_2(1-u))}{t_2 u t_2 (1-u)}. \end{aligned}$$

Defining

$$K(t_2) \equiv \frac{\sqrt{3} t_2/\pi}{\sinh(t_2)/c_2(t_2)} > 0 \quad \text{for } t_2 > 0,$$

then – analogue to case 2, but now using hyperbolic functions – for $t_2 > 0$,

$$\begin{aligned} A'(u) = K(t_2) t_2 \cdot \frac{\sinh(t_2 u)\sinh(t_2(1-u))}{t_2 u t_2(1-u)} \cdot \Big(\Big[\coth(t_2 u) - 1/(t_2 u) \Big] \\ - \Big[\coth(t_2(1-u)) - 1/(t_2(1-u)) \Big] \Big) \end{aligned}$$

Now define $z(x) \equiv \coth(x) - 1/x$ which is strictly monotone increasing for $x > 0$, because by means of $\sinh(x) > x$ for $x > 0$,

$$z'(x) = -\frac{1}{(\sinh(x))^2} + \frac{1}{x^2} > 0.$$

Hence, from $t_2(1-u) < t_2 u$ for $1/2 < u < 1$ and $t_2 > 0$ follows

$$\Big[\coth(t_2 u) - 1/(t_2 u) \Big] - \Big[\coth(t_2(1-u)) - 1/(t_2(1-u)) \Big] > 0,$$

This completes the proof of case 3.

Case 4: Assume $t_1 > 0$ and $t_2 > t_1$.

Similar to case 1 we have for $0 < u < 1$

$$A(u) = \frac{t_1/c_2(t_1)}{t_2/c_2(t_2)} \cdot \frac{\frac{\sinh(t_1)}{\sinh(t_1 u)\sin(t_1(1-u))}}{\frac{\sinh(t_2)}{\sin(t_2 u)\sinh(t_2(1-u))}} = \frac{t_1/c_2(t_1)\sinh(t_1)}{t_2/c_2(t_2)\sinh(t_2)} \cdot \frac{\sinh(t_2 u)\sinh(t_2(1-u))}{\sinh(t_1 u)\sinh(t_1(1-u))}.$$

Defining

$$K(t_1, t_2) \equiv \frac{t_1/c_2(t_1)}{t_2/c_2(t_2)} \frac{\sinh(t_1)}{\sinh(t_2)} > 0,$$

the first derivative of A is given by

$$A'(u) = K(t_1, t_2) \cdot \frac{\sinh(t_2 u)\sinh(t_2(1-u))}{\sinh(t_1 u)\sinh(t_1(1-u))} \cdot \Big[t_2(\coth(t_2 u) - \coth(t_2(1-u))) - t_1(\coth(t_1 u) - \coth(t_1(1-u)))\Big].$$

Now,

$$z(t) = \coth(tu) - \coth(t(1-u)) \quad \text{is stictly monotone increasing for } t > 0, \qquad (3)$$

if the first derivative $z'(t)$ is positive: For $t > 0,\ 1/2 < u < 1$,

$$\frac{1-u}{[\sinh(t(1-u)))]^2} - \frac{u}{[\sinh(tu)]^2} > 0 \qquad \Longleftrightarrow$$

$$\frac{t(1-u)}{[\sinh(t(1-u)))]^2} - \frac{tu}{[\sinh(tu)]^2} > 0 \qquad \Longleftrightarrow$$

$$1 - \frac{tu}{t(1-u)} \frac{[\sinh(t(1-u))]^2}{[\sinh(tu)]^2} > 0 \qquad \Longleftrightarrow \qquad \frac{\frac{[\sinh(t(1-u))]^2}{t(1-u)}}{\frac{[\sinh(tu)]^2}{tu}} < 1.$$

To prove the last inequality, we show that $f(x) = \frac{[\sinh(x)]^2}{x}$ is monotone increasing for $x > 0$. Using $x \geq \tanh(x)$,

$$f'(x) = \frac{2\sinh(x)\cosh(x)x - [\sinh(x)]^2}{x^2} = \frac{[\cosh(x)]^2}{x^2}\left(2\tanh(x)x - [\tanh(x)]^2\right) \geq \frac{[\cosh(x)]^2}{x^2}\tanh(x)x \;\geq\; 0.$$

From equation (3) follows that

$$\coth(t_2 u) - \coth(t_2(1-u)) > \coth(t_1 u) - \coth(t_1(1-u))$$

and

$$t_2\Big[\coth(t_2 u) - \coth(t_2(1-u))\Big] - t_1\Big[\coth(t_1 u) - \coth(t_1(1-u))\Big] > 0.$$

Consequently, $A'(u) > 0$ for $1/2 < u < 1$. □

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