Friedrich-Alexander-Universität Erlangen-Nürnberg

Wirtschafts-und Sozialwissenschaftliche Fakultät

Diskussionspapier

52 / 2003

Tukey-type distributions in the context of financial data

Matthias Fischer, Armin Horn and Ingo Klein

Lehrstuhl für Statistik und Ökonometrie

Lehrstuhl für Statistik und empirische Wirtschaftsforschung

Lange Gasse 20 · D-90403 Nürnberg

TUKEY-TYPE DISTRIBUTIONS IN THE CONTEXT OF FINANCIAL DATA

Matthias Fischer, Armin Horn and Ingo Klein

Department of Statistics and Econometrics

University of Erlangen-Nuremberg

Nuremberg, Germany

Key Words: kurtosis; skewness; variable transformation; transformed Gaussian; return data.

ABSTRACT

Using the Gaussian distribution as statistical model for data sets is widely spread, especially in practice. However, departure from normality seems to be more the rule than the exception. The H-distributions, introduced by Tukey (1960, 1977), are generated by a single transformation (H-transformation) of a standard normal distribution (or, more general, of a symmetric distribution) Z and allow for leptokurtosis represented by the (elongation) parameter $h > 0$. In order to additionally take skewness into account by means of certain transformations, several generalizations and extensions $(HQ, HH, GH, GK, \ldots)$ have been proposed in the literature. Within this work we "complete" this class of Tukey-type distributions by introducing KQ- and JQ-distributions on the one side and $KK-$, $JJ-$ and $\widetilde{G}J-$distributions on the other side. Moreover, we empirically compare the goodness-of-fit of such Tukey-type distributions for different symmetrical distributions Z (here: Gaussian, logistic and hyperbolic secant distribution) in the context of financial return data. In particular, the interplay between Z and the Tukey-type transformations is investigated. Finally, results are compared to those of popular multi-parametric distribution models with closed-form densities.

1. INTRODUCTION

Using the Gaussian distribution as statistical model for data sets is widely spread, especially in practice. However, departure from normality seems to be more the rule than the exception. The H-family of distributions or H-distributions, introduced by Tukey (1960, 1977), are generated by a single transformation of the standard normal distribution Z and allow for leptokurtosis represented by the (elongation) parameter $h > 0$. More precisely, H-distributions are asymptotically Pareto-heavy tailed with tail index $1/h$ which implies that only moments of order less than $1/h$ exist. The degree of elongation can be further increased if the parameter h is additionally allowed to be a function of $Z^2, Z^4, ...$ If h is only a function of Z^2, the corresponding distribution is commonly termed as HQ-distribution (cf., Morgenthaler and Tukey, 2000), where the second parameter Q quantifies the influence of Z^2. Alternatively, Haynes et al. (1997) proposed the K-distribution which is also heavy-tailed but for which all moments exist. Fischer and Klein (2003) additionally suggested the J-distribution which lies somewhere between both distributions in the sense that tails are heavier than that of the K-distribution but moments still exist.

Typically, leptokurtic data sets also exhibit a certain amount of skewness. To capture this phenomenon, the above-mentioned kurtosis transformations can be combined with skewness transformations as, for example, the G-transformation of Tukey (1977) or the generalized $\widetilde{G}$-transformation of Haynes et al. (1997). The resulting GH-distribution, GK-distribution and $\widetilde{G}K$-distribution have been intensively studied in the literature (see Hoaglin, 1983, Martinez and Iglewicz, 1984 or MacGillivray, 1981. Applications with respect to financial return data are given by Badrinath and Chatterjee (1988, 1989) who apply GH-distributions to the returns of stock market indices and to several US-equities. Mills (1995) demonstrates the "excellent fit of the GH-distributions" to the distribution of the daily returns on the London Stock Exchange FT-SE indices). Another possibility to additionally take skewness into account is to "double" the transformation, i.e. to introduce a kurtosis parameter for the positive and the negative part of the axis. This idea goes back to Morgenthaler and Tukey (2000) who introduced the so-called HH-distribution.

Within this work we "complete" the so-called Tukey-type distributions from above, i.e. we introduce KQ- and JQ-distributions on the one side and $QQ-$, $JJ-$ and $\widetilde{G}J-$distributions on the other side. Moreover, we empirically analyze the goodness-of-fit of such Tukey-type distributions for different distributions Z (Gaussian, logistic and hyperbolic secant). In particular, the interplay between Z and the transformations is investigated. Finally, results are compared to those of popular distribution models with closed-form densities like generalized t-distributions (see Theodossiou, 1998 or Grottke, 2001), generalized logistic distributions (see McDonald, 1991 or Fischer, 2001), generalized hyperbolic secant distributions (see Fischer and Vaughan, 2002) and generalized hyperbolic distributions (see Prause, 1999).

2. KURTOSIS AND SKEWNESS TRANSFORMATION: REQUIREMENTS AND EXAMPLES

2.1 Generating leptokurtosis

Let Z be a random variable which is symmetric around the median 0 and which has continuous distribution function. Define

$$X \equiv \mu + \delta \mathrm{T}(Z) = \mu + \delta Z \cdot \mathrm{W}(Z), \quad \mu \in \mathbb{R}, \delta > 0 \tag{1}$$

where T is a suitable kurtosis transformation. Hoaglin (1983) postulated some plausible requirements to T, that is

K1 *Symmetry*: If $\mathrm{W}(z) = \mathrm{W}(-z)$ for $z \in \mathbb{R}$, i.e. W is preserving symmetry and we can restrict discussion to the positive axis.

K2 *Invariance in the center*: The initial distribution T should hardly be transformed in the center, i.e. $\mathrm{T}(z) \approx z$ for $z \approx 0$.

K3 *Smoothness*: $\mathrm{T}(z)$ should be a sufficiently smooth function with continuous second derivative.

K4 *Tail elongation*: To assure that T is accelerated strictly monotone increasing for positive $z > 0$, i.e. $\mathrm{T}'(z) > 0$ and $\mathrm{T}''(z) > 0$ for $z > 0$. Consequently, T is strictly monotone increasing and convex for $z > 0$.

Example 1 ($H-$, $K-$ and $J-$transformation) *The $H-$transformation*

$$\mathrm{T}_h(z) \equiv z \exp(hz^2/2), \quad h \geq 0$$

was proposed by Tukey (1960). The K-transformation of Haynes et al. (1998) is given by

$$\mathrm{T}_k(z) \equiv z(1+z^2)^k, \quad k \geq 0$$

and the J-transformation of Fischer and Klein (2003) by

$$\mathrm{T}_j(z) \equiv z \cosh(z)^j = \frac{z}{2}\left(\exp(z) + \exp(-z)\right)^j, \quad j \geq 0.$$

Note that all tree transformations are nested in the so-called kurtosis power transformation of Klein and Fischer (2003) which is given by

$$T_r(z) = zA(z) \ \text{ with } \ A(z) \equiv \left(\sum_{i=0}^{\infty} a_i z^{2i}\right)^r \quad \text{for } r \geq 0,$$

with specific weights a_i.

Example 2 ($HQ-$, $KQ-$ and $JQ-$transformation) *In order to further increase the tails, Morgenthaler and Tukey (2000) added the term $qz^4/4$ to the exponent (this is equivalent to allow h to be a function of z^2, i.e. setting $h(z) = h + 0.5qz^2$). The corresponding transformation was termed as HQ-transformation*

$$\mathrm{T}_{h,q}(z) \equiv z \exp(hz^2/2 + qz^4/4), \quad h, q \geq 0.$$

Similarly, we define the KQ-transformation by

$$\mathrm{T}_{k,q}(z) \equiv z(1+z^2)^{k+qz^2}, \quad k, q \geq 0$$

and the JQ-transformation by

$$\mathrm{T}_{j,q}(z) \equiv z(\cosh(z))^{j+qz^2}, \quad j, q \geq 0$$

Clearly, for $q = 0$, the transformations of examples 1 are obtained.

Lemma 1 *The first derivatives of the $HQ-$, the $JQ-$ and the $KQ-$transformations are given by*

$$\mathrm{T}'_{h,q}(z) = \mathrm{T}_{h,q}(z)(1/z + hz + qz^3),$$

$$\mathrm{T}'_{j,q}(z) = \mathrm{T}_{j,q}(z)\left(1/z + j\tanh(z) + q\left(2\,z\ln(\cosh(z)) + \tanh(z)\,z^2\right)\right)$$

and

$$\mathrm{T}'_{k,q}(z) = \mathrm{T}_{k,q}(z)\left(1/z + \frac{2zk}{1+z^2} + q\left(2z\ln\left(1+z^2\right) + \frac{2\,z^3}{1+z^2}\right)\right).$$

The strength of the tail-elongation of a transformation can be determined by the so-called elongation generating function (EGF) $f(x)$ which was introduced by Fischer and Klein (2003) as a C^2-function living on the real line with $f(-x) = -f(x), f(x) > 0$ for $x > 0$ and $x\frac{f'(x)}{f(x)} \geq -2$ for $x > 0$. The EGF is linear for the $H-$transformation, asymptotically constant for the $J-$transformation and asymptotically zero for the $K-$transformation ("The higher the slope of the EGF the higher the elongation"). Its relation to the kurtosis transformation is given by

$$T_{\theta,f}(z) = z\exp\left(\theta\int_0^z f(u)\,du\right). \tag{2}$$

Using lemma 1 we can calculate the elongation generating functions $f(z)$ for the generalized transformations of example 2.

Lemma 2 *The elongation generating functions for the $HQ-$, $JQ-$ and $KQ-$transformations are given by*

$$f_{HQ}(z;q^*) = z + q^*z^3 \qquad \text{("cubic EGF")}$$

$$f_{JQ}(z;q^*) = \tanh(z) + q^*\left(2\,z\ln(\cosh(z)) + \tanh(z)\,z^2\right) \qquad \text{("quadratic-type EGF")}$$

$$f_{KQ}(z;q^*) = \frac{z}{1+z^2} + q^*\cdot\frac{2z\left((1+z^2)\ln(1+z^2) + z^2\right)}{1+z^2} \qquad \text{("linear-type EGF"),}$$

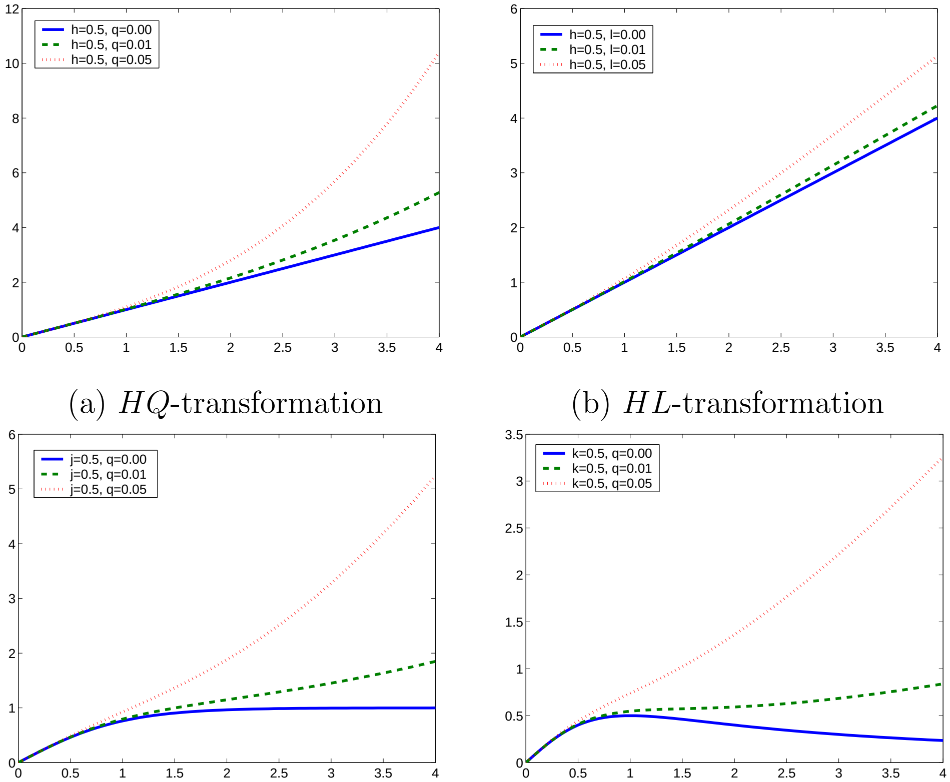
where $q^* = q/\theta$. Note that setting $q \equiv 0$ results in the EGF's of the standard $H-, J-$ and $K-$transformation.

To proof lemma 1, you have to solve equation (2) for $f(z)$, i.e.

$$f(z) = \frac{1}{\theta}\left(\frac{\mathrm{T}'_{f,\theta}(z)}{\mathrm{T}_{f,\theta}(z)} - \frac{1}{z}\right).$$

Typical plots of such EGF's are given in figure 1, below.

Figure 1: EGF for different transformations



(a) HQ-transformation (b) HL-transformation

(c) JQ-transformation (d) KQ-transformation

Finally the H-transformation can be generalized by generalizing its EGF to a function of type "$x \log(x)$" (i.e. which produces heavier tails than the $H-$transformation but lighter than the $JQ-$ and the $HQ-$transformation):

$$f_{HL}(z; l^*) \equiv z(1 + l^* \log(1 + z^2)), \quad z \in \mathbb{R},\ l^* = l/h > 0. \tag{3}$$

Clearly, for $l = 0$, equation (3) reduces to the EGF of the H-transformation. Applying formula (2), the corresponding transformation ("HL-transformation") is given by

$$\mathrm{T}_{h,l}(z) \equiv z \left(1 + z^2\right)^{l/2(1+z^2)} e^{0.5\, z^2 (h-l)}.$$

with first derivation given by

$$\mathrm{T}'_{h,l}(z) = \mathrm{T}_{h,l}(z) \left(1/z + l \ln\left(z + z^2\right) z + zh\right).$$

2.2 Generating kurtosis and skewness

Hoaglin's requirements to a skewness transformation are the following:

S1 *Invariance in the center*: the initial distribution T should hardly be transformed in the center, i.e. $\mathrm{T}(0) = 0$, $\mathrm{T}(z) \approx z$ for $z \approx 0$.

S2 *Smoothness*: $\mathrm{T}(z)$ should be a sufficiently smooth function with continuous second derivative.

S3 *Monotony*: T has to be strictly increasing and convex ($\mathrm{T}'(z) > 0$ and $\mathrm{T}''(z) > 0$), or strictly increasing and concave ($\mathrm{T}'(z) > 0$ and $\mathrm{T}'(z) < 0$). If T is strictly increasing and convex, $\mathrm{T}'(0) = 1$ implies that $\mathrm{T}'(z) > 1$ for $z > 0$ and $0 < \mathrm{T}'(z) < 1$ for $z < 0$. This means that the left tails of X are shortened and the right tails are made longer (i.e. an increase of the skewness to the right).

Example 3 (G- and $\widetilde{G}$-transformation) *The G-transformation of Tukey (1960) is of the form*

$$\mathrm{T}_g(z) = z\left(\frac{e^{gz} - 1}{g}\right) \qquad \textit{for } g \in \mathbb{R}. \tag{4}$$

Obviously, $\mathrm{T}_g(z) > -1/g$ if $g > 0$, and $\mathrm{T}_g(z) < -1/g$ if $g < 0$. Consequently, the range of $\mathrm{T}_g(z)$ is restricted one-sided. This is a desirable property at least for left-skewed return distributions because losses larger than 100 percent are impossible. When g converges to zero, $\mathrm{T}_g(z) \to z$, i.e. X and Z coincide. A transformation with nearly identical fit but which is defined on the whole real line is the generalized G or – in our notation – the $\widetilde{G}$-transformation of Haynes, MacGillivray and Mengersen (1997)

$$\mathrm{T}_{\widetilde{g}}(z) = z\left(1 + c \cdot \frac{1 - e^{-\widetilde{g}z}}{1 + e^{-\widetilde{g}z}}\right) \qquad \textit{for } g \in \mathbb{R}. \tag{5}$$

where c is typically set to 0.8.

Combining the skewness transformations from example 3 and the kurtosis transformation from example 1 via

$$\mathrm{T}(z) \equiv \mathrm{T}_{Skew}(z)/z\mathrm{T}_{Kurt}(z) \tag{6}$$

enables us to generate flexible asymmetric heavy-tailed distributions.

"Doubling" is another method to additionally take skewness into account. Morgenthaler and Tukey (2000) chose a pair (h_1, h_2) of positive constants to transform separately for $Z \leq 0$ and $Z \geq 0$. The resulting transformation

$$\mathrm{T}_{h_1,h_2}(z) \equiv \begin{cases} z\exp(h_1 z^2/2) & \text{for } z \leq 0 \\ z\exp(h_2 z^2/2) & \text{for } z \geq 0 \end{cases} \tag{7}$$

is called the $HH-$transformation. It is is straightforward to transform this idea to both the $K-$ and the $J-$transformation. So we define the $KK-$transformation by

$$\mathrm{T}_{k_1,k_2}(z) \equiv \begin{cases} z(1+z^2)^{k_1} & \text{for } z \leq 0 \\ z(1+z^2)^{k_2} & \text{for } z \geq 0 \end{cases} \tag{8}$$

and the $JJ-$transformation by

$$\mathrm{T}_{j_1,j_2}(z) \equiv \begin{cases} z\cosh(z)^{j_1} & \text{for } z \leq 0 \\ z\cosh(z)^{j_2} & \text{for } z \geq 0. \end{cases} \tag{9}$$

3. TRANSFORMATION OF SYMMETRIC DISTRIBUTIONS

Let Z denote an arbitrary symmetrical distribution and $\mathrm{T}_\theta(Z)$ one of the Tukey-type transformation which have been discussed in the last section. Then we can generate a random variable X by means of

$$X \equiv \mu + \delta \mathrm{T}_\theta(Z), \quad \mu \in \mathbb{R}, \delta > 0$$

which allows for skewness and/or (excess) kurtosis. The probability density function f_X and the quantile function of such a Tukey-type distribution can be calculated via variable transformation as stated in the next proposition:

Proposition 1 (Density and quantiles of X) *Let T_θ denote an arbitrary Tukey-type transformation and T_θ^{-1} the inverse mapping of T_θ.*

1. *The probability density function of X is given by*

$$f_X(x;\mu,\sigma,\theta) = \frac{f_Z(\mathrm{T}_\theta^{-1}(\frac{x-\mu}{\sigma}))}{\mathrm{T}'_\theta(\mathrm{T}_\theta^{-1}(\frac{x-\mu}{\sigma}))}.$$

2. *The p-quantiles of X can be obtained from the p-quantiles of Z by means of*

$$x_p = \mu + \sigma \cdot \mathrm{T}_\theta(z_p). \tag{10}$$

4. APPLICATION TO FINANCIAL RETURN DATA

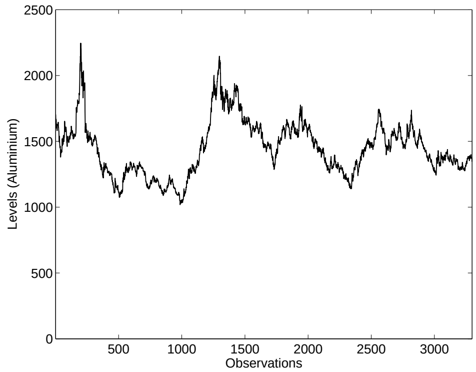
In order to compare Tukey-type distributions towards their fit, we focus on the series of aluminium and zinc from January 1990 to December 2002 ($N = 3279$ observations for each series) which can be obtained from the London Metal Exchange [1] (LME) the world's premier non-ferrous metals market. The series of prices and corresponding log-returns is given in figure 1, below.

The (sample) mean of the log-returns of aluminium is -0.0067 with a (sample) standard deviation of 1.1986. Moreover, there seems to be no remarkable skewness in the data set (the skewness coefficient – measured by the third standardized moments – is given by by 0.0075), whereas the kurtosis coefficient – in terms of the fourth standardized moments – is 8.00, reflecting the high leptokurtosis of the data. This is the reason why we first apply and compare only kurtosis transformations to different symmetric distribution (i.e. Gaussian, logistic and hyperbolic secant distribution).

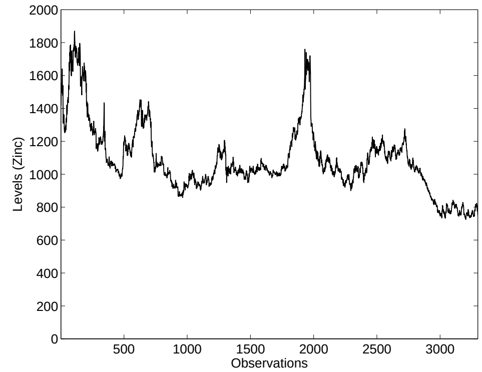
On the other hand, the returns of zinc with sample mean of -0.0001929 and standard deviation of 0.0141 has a skewness coefficient of -0.94 with a high kurtosis of 14.07. Even if we remove the three smallest values, skewness coefficient is about -0.30 and kurtosis coefficient about 7.56. This data set is chosen to compare the goodness-of-fit of transformations which also take skewness into account (i.e. $\widetilde{G}H-$, $HH-$, $\widetilde{G}J-$, $JJ-$, $KK-$ and $\widetilde{G}K-$transformation) .

[1] Download under `http://www.lme.co.uk/`.

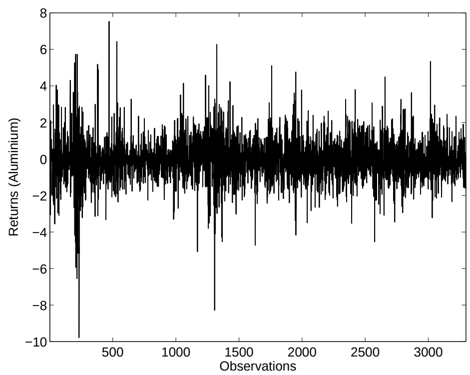
Figure 2: Prices and log-returns



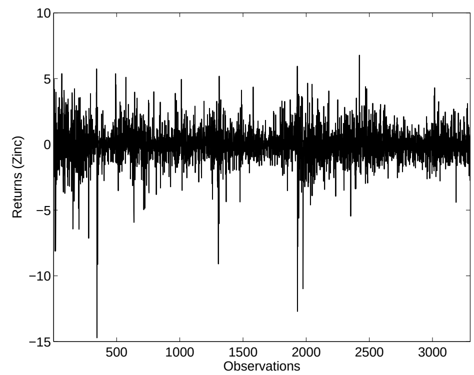
(a) Aluminium prices



(b) Zinc prices



(c) Aluminium returns



(d) Zinc returns

Four criteria have been employed to compare the goodness-of-fit of the different candidate distributions. The first is the *log-Likelihood value* ($\mathcal{LL}$) obtained from the Maximum-Likelihood estimation. The $\mathcal{LL}$-value can be considered as an "overall measure of goodness-of-fit and allows us to judge which candidate is more likely to have generated the data". As distributions with different numbers of parameters k are used, this is taken into account by calculating the *Akaike criterion* given by

$$AIC = -2 \cdot \mathcal{LL} + \frac{2N(k+1)}{N-k-2}.$$

The third criterion is the *Kolmogorov-Smirnov distance* as a measure of the distance between the estimated parametric cumulative distribution function, $\hat{F}$, and the empirical sample distribution, F_{emp}. It is usually defined by

$$\mathcal{K} = 100 \cdot \sup_{x \in \mathbb{R}} |F_{emp}(x) - \hat{F}(x)|. \tag{11}$$

Finally, *Anderson-Darling statistic* is calculated, which weights $|F_{emp}(x) - \hat{F}(x)|$ by the reciprocal of the standard deviation of F_{emp}, namely $\sqrt{\hat{F}(x)(1-\hat{F}(x))}$, that is

$$\mathcal{AD}_0 = \sup_{x \in \mathbb{R}} \frac{|F_{emp}(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}}. \tag{12}$$

Instead of just the maximum discrepancy, the second and third largest value, which is commonly termed as $\mathcal{AD}_1$ and $\mathcal{AD}_2$, are also taken into consideration. Whereas $\mathcal{K}$ emphasizes deviations around the median of the fitted distribution, $\mathcal{AD}_0, \mathcal{AD}_1$ and $\mathcal{AD}_2$ allow discrepancies in the tails of the distribution to be appropriately weighted. The estimation results are summarized in table 1 to table 5.

What are the major drawbacks? Let's first focus on the symmetric case. Firstly, there is a trade-off between the kurtosis parameter of the transformation and the amount of kurtosis of the underlying distribution (i.e. which is to be transformed). The higher the kurtosis of the underlying distribution the lower the value of the kurtosis parameter. It should be pointed out that the combination of the hyperbolic secant distribution (which has higher kurtosis than the logistic or the normal distribution) with an arbitrary kurtosis transformation fits worse than the corresponding combinations of the normal and the logistic distribution. In general, there is no need or improvement to start with a distribution which is more leptokurtic than the normal distribution, unless the $j-$ or the $k-$transformation is used. Secondly, if we only consider the one-parameter-transformations, the $H-$distribution dominates the other competitors. Tails of the $j-$distribution and the $k-$distribution seem to be too "moderate". Considering the corresponding two-parameter transformation this result is no longer valid: There is seemingly no difference (concerning the goodness-of-fit) between the $HQ-$, $KQ-$, $JQ-$ or $HL-$distribution. Tails of the underlying return data are best approximated (in terms of the $\mathcal{AD}$-statistics) by EGF's of the type "$x\log(x)$", i.e. by the $HL-$distribution.

Moreover, comparing table 1 with table 4, symmetric Tukey-type distributions exhibit nearly identical goodness-of-fit results than the generalized t-distribution (GT) and better results than popular parametric distribution models like the symmetric generalized logistic (EGB2) distribution (see McDonald, 1991), the symmetric generalized hyperbolic secant (GSH, GHS) distribution (see Fischer and Vaughan, 2002) or the symmetric generalized hyperbolic (GH) distribution (see Prause, 1999) which have been proposed in the context of financial return data in the recent literature. Consequently, the results support the hypothesis that moments of financial return data only exist up to a certain order.

The results of the asymmetric case are similar. However, $GH-$ and $HH-$distribution now outperform the corresponding versions based on the $K-$ and the $J-$transformations. In particular, $GK-$ and $KK-$distribution are not so promising. Whether skewness is introduced via doubling or via combination of skewness and kurtosis transformation has no remarkable effect on the estimation results. Finally, skew Tukey-type distributions dominate the corresponding parametric distribution with closed-form densities except the skewed generalized t-distribution (SGT2) of the second kind (see Grottke 2001).

5. SUMMARY

Tukey-type distributions are generated by a single transformation of the standard normal (or, more general, of any symmetric) distribution. These distributions are able to model leptokurtic and/or skew data. By means of elongation generating functions the strength of tail elongation of these Tukey-type distributions can be compared. Within this work we reviewed Tukey-type distributions which have been proposed in the literature up to now. Moreover, we proposed alternative Tukey-type distributions corresponding to alternative Tukey-type transformations. Finally, we are empirically investigated the goodness-of-fit of such distribution families in the context of financial return data most which exhibit high kurtosis and a certain amount of skewness. In particular, goodness-of-fit measures were compared to that of popular parametric distribution models whose density is given in closed form. The main results with respect to the underlying return data are:

- It is not necessary to start with a distribution which is more leptokurtic than the normal distribution.
- The $H-$transformation dominates the other one-parameter kurtosis transformations.
- There is no significant difference between the two-parameter kurtosis transformations which have been considered.
- Introducing skewness by doubling or by combination of skewness and kurtosis transformations makes no difference.
- Combining the $H/J-$transformation with the $G-$transformation seems more promising than combining the $K-$transformations with it.
- There is a high similarity between the fit of Tukey-type distributions and that of (skewed) generalized t-distributions. Moreover, Tukey-type distributions dominated distribution families with closed-form densities and existing moments like the generalized logistic family, the generalized hyperbolic family or the generalized hyperbolic secant family.

Table 1: Goodness-of-fit and estimated parameters: **Symmetric case**

Type	$\mathcal{LL}$	$\mathcal{AIC}$	$\mathcal{KS}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\widehat{\mu}$	$\widehat{\delta}$	$\widehat{h}/\widehat{j}/\widehat{k}$	$\widehat{q},\widehat{l}$
Transformed Gaussian (N)										
N	-5273.3	10552.7	5.17	23202	24.047	19.189	-0.007	1.44	(0.000)	(0.000)
H	-5051.9	**10111.8**	**1.08**	0.043	0.042	0.041	-0.028	0.92	0.149	(0.000)
J	-5059.1	10126.3	1.37	0.087	0.083	0.057	-0.032	0.88	0.271	(0.000)
K	-5077.9	10163.7	1.73	0.351	0.269	0.092	-0.034	0.84	0.265	(0.000)
HL	**-5051.4**	10112.8	1.15	**0.038**	**0.038**	**0.037**	-0.027	0.93	0.107	0.031
HQ	-5051.6	10113.2	1.12	0.039	**0.038**	**0.037**	-0.028	0.93	0.130	0.006
JQ	-5051.6	10113.3	1.13	0.039	0.038	0.038	-0.027	0.93	0.131	0.020
KQ	-5051.5	10112.9	1.14	0.039	0.038	0.038	-0.027	0.93	0.064	0.027
Transformed logistic (L)										
L	-5085.0	10176.1	2.50	0.771	0.465	0.169	-0.021	0.39	(0.000)	(0.000)
H	**-5051.7**	**10111.4**	1.11	0.040	0.039	0.038	-0.028	1.05	0.057	(0.000)
J	-5053.3	10114.7	1.19	0.048	0.046	0.046	-0.029	1.03	0.112	(0.000)
K	-5058.4	10124.8	1.27	0.099	0.090	0.054	-0.031	1.01	0.109	(0.000)
HL	**-5051.7**	10113.4	1.11	0.040	0.039	0.038	-0.028	1.05	0.057	0.001
HQ	**-5051.7**	10113.4	1.11	0.040	0.039	0.038	-0.028	1.05	0.057	(0.000)
JQ	-5052.1	10114.2	**1.10**	0.040	0.039	0.039	-0.028	1.05	0.069	0.005
KQ	-5051.9	10113.9	1.12	**0.039**	**0.038**	**0.038**	-0.027	1.05	0.032	0.009
Transformed hyperbolic cosine (HC)										
HC	-5064.4	10134.8	1.27	0.282	0.203	0.087	-0.028	1.34	(0.000)	(0.000)
H	-5054.3	10116.5	1.35	0.051	0.050	0.049	-0.030	1.12	0.026	(0.000)
J	-5056.7	10121.4	1.41	0.061	0.059	0.055	-0.031	1.11	0.051	(0.000)
K	-5060.1	10128.2	1.34	0.123	0.104	0.058	-0.031	1.11	0.043	(0.000)
HL	**-5053.1**	**10116.2**	1.22	0.046	0.045	0.044	-0.029	1.13	0.000	0.015
HQ	-5053.9	10117.9	1.23	0.047	0.046	0.045	-0.030	1.13	0.012	0.002
JQ	-5053.3	10116.7	**1.19**	**0.046**	**0.044**	**0.044**	-0.029	1.13	0.001*	0.005
KQ	-5053.2	10116.4	1.25	0.047	0.046	0.045	-0.030	1.13	0.001*	0.005

Table 2: Goodness-of-fit and estimated parameters: **Asymmetric case (I)**

Type	$\mathcal{LL}$	$\mathcal{AIC}$	$\mathcal{KS}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\widehat{\mu}$	$\widehat{\delta}$	$\widehat{h}/\widehat{j}/\widehat{k}$	$\widehat{q}$
Transformed Gaussian (G)										
N	-5818.3	11642.7	7.05	>100	47.95	38.37	-0.019	1.413	(0.000)	(0.000)
$\widetilde{G}$	-5783.5	11574.7	7.61	>100	3271	172.8	0.015	1.399	-0.061	(0.000)
H	-5462.0	10932.1	1.40	0.057	0.056	0.056	-0.035	0.985	(0.000)	0.193
J	-5466.2	10940.4	1.83	0.133	0.118	0.117	-0.040	0.936	(0.000)	0.347
K	-5487.1	10982.1	2.45	1.105	0.672	0.416	-0.044	0.877	(0.000)	0.342
$\widetilde{G}$H	**-5459.8**	**10929.6**	**1.04**	0.064	0.064	0.061	-0.038	0.983	0.063	0.195
$\widetilde{G}$J	-5465.9	10941.8	1.66	0.149	0.135	0.130	-0.040	0.935	0.021	0.349
$\widetilde{G}$K	-5486.2	10982.3	2.70	0.721	0.484	0.316	-0.043	0.880	-0.031	0.339
HH	-5461.9	10933.9	1.36	0.053	**0.052**	**0.052**	**-0.035**	0.984	0.189	0.199
JJ	-5466.2	10942.4	1.86	0.128	0.114	0.113	-0.040	0.936	0.350	0.344
KK	-5486.5	10983.1	2.67	0.810	0.524	0.336	-0.044	0.879	0.353	0.327
Transformed logistic (L)										
L	-5534.6	11075.3	4.00	9.852	4.768	2.117	-0.022	0.708	(0.000)	(0.000)
$\widetilde{G}$	-5531.8	11071.6	3.39	3.023	1.837	0.968	-0.017	1.284	-0.028	(0.000)
H	-5463.4	10934.9	1.43	0.064	0.062	0.062	-0.033	1.139	(0.000)	0.089
J	-5462.1	10932.3	1.52	0.061	0.059	0.059	-0.037	1.103	(0.000)	0.176
K	-5467.0	10942.0	1.85	0.161	0.157	0.135	-0.040	1.063	(0.000)	0.178
$\widetilde{G}$H	**-5461.1**	**10932.2**	**1.16**	**0.054**	**0.052**	**0.051**	-0.036	1.136	0.064	0.092
$\widetilde{G}$J	-5461.5	10933.1	1.28	0.071	0.070	0.070	-0.038	1.101	0.031	0.179
$\widetilde{G}$K	-5466.9	10943.9	1.91	0.150	0.143	0.126	-0.039	1.064	-0.008	0.177
HH	-5463.4	10936.9	1.43	0.064	0.063	0.062	-0.033	1.139	0.089	0.088
JJ	-5462.1	10934.3	3.39	0.068	0.061	0.059	-0.037	1.103	0.175	0.177
KK	-5467.0	10944.0	1.82	0.166	0.162	0.138	-0.040	1.063	0.176	0.180

Table 3: Goodness-of-fit and estimated parameters: **Asymmetric case (II)**

Type	$\mathcal{LL}$	$\mathcal{AIC}$	$\mathcal{KS}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\widehat{\mu}$	$\widehat{\delta}$	$\widehat{h}/\widehat{j}/\widehat{k}$	$\widehat{q}$
Transformed hyperbolic secant (HS)										
HS	-5493.7	10993.5	1.94	2.558	1.535	0.817	-0.032	1.308	(0.000)	(0.000)
$\widetilde{G}$	-5491.1	10990.3	2.16	0.912	0.673	0.419	-0.029	1.308	-0.032	(0.000)
H	-5461.6	**10931.2**	1.47	**0.056**	**0.054**	**0.054**	-0.036	1.224	(0.000)	0.048
J	-5463.3	10934.6	1.64	0.080	0.076	0.071	-0.039	1.200	(0.000)	0.104
K	-5468.3	10944.7	1.88	0.197	0.189	0.151	-0.041	1.176	(0.000)	0.107
$\widetilde{G}$H	**-5460.6**	10931.2	**1.17**	0.061	0.061	0.057	-0.037	1.220	0.042	0.051
$\widetilde{G}$J	-5463.2	10936.5	1.57	0.085	0.080	0.074	-0.039	1.198	0.008	0.105
$\widetilde{G}$K	-5467.8	10945.7	2.04	0.150	0.147	0.124	-0.040	1.178	-0.022	0.104
HH	-5460.8	10931.6	1.64	0.066	0.064	0.064	-0.036	1.230	0.056	0.033
JJ	-5462.9	10935.8	1.78	0.067	0.064	0.063	-0.039	1.203	0.114	0.088
KK	-5468.2	10946.5	1.96	0.178	0.173	0.139	-0.041	1.177	0.112	0.101

Table 4: Goodness-of-fit: **Symmetric case**

Type	$\mathcal{LL}$	$\mathcal{AIC}$	$\mathcal{KS}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
Alternative symmetric distribution models						
GT	**-5051.5**	10113.0	1.15	**0.038**	**0.038**	**0.038**
GH	-5051.5	10113.0	**1.10**	0.041	0.039	0.039
GSH	-5061.7	10131.4	1.28	0.174	0.136	0.061
GHS	-5060.9	10129.8	1.29	0.149	0.120	0.058
EGB2	-5063.4	10134.8	1.29	0.231	0.172	0.068

Table 5: Goodness-of-fit: **Asymmetric case**

Type	$\mathcal{LL}$	$\mathcal{AIC}$	$\mathcal{KS}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
Alternative asymmetric distribution models						
SGT2	**-5457.9**	**10927.8**	**0.98**	**0.064**	**0.064**	**0.060**
GH	-5461.8	10935.6	1.23	0.064	0.063	0.061
SGSH	-5471.1	10952.3	1.55	0.560	0.429	0.280
SGHS	-5469.5	10949.1	1.54	0.414	0.341	0.236
EGB2	-5479.2	10968.5	1.77	1.482	0.967	0.553

BIBLIOGRAPHY

Badrinath, S. G. and S. Chatterjee (1988): On Measuring Skewness and Elongation in common Stock Return Distributions: The Case of the Market Index. *Journal of Business*, 61:451-472.

Badrinath, S. G. and S. Chatterjee (1991): A data-analytic look at Skewness and Elongation in common Stock Return Distributions. *Journal of Business and Economics Statistics*, 9:223-233.

Fischer, M. (2002): *Infinitely Divisible Distribution and its Application towards Financial Return Data: Unconditional Fit and Option Pricing.* Quantitative Finanzwirtschaft Band Nr. 2, in: S. Mittnik (ed.), Schriftenreihe zu Statistik und Ökonometrie.

Fischer, M. and I. Klein (2003): Kurtosis modelling by means of the j-transformation. *Diskussion paper 52*, University of Erlangen-Nürnberg (submitted).

Fischer, M. and D. Vaughan (2002): Classes of Skewed Generalized Secant Hyperbolic Distributions. *Diskussionspaper Nr. 45*, Lehrstuhl für Statistik und Ökonometrie, Universität

Erlangen-Nürnberg, (submitted).

Grottke, M. (2001): *Die t-Verteilung und ihre Verallgemeinerungen als Modell für Finanzmarktdaten.* Josef Eul Verlag, Lothmar.

Haynes, M. A. and H. L. MacGillivray and K. L. Mengersen (1997): Robustness of ranking and selection rules using generalized g-and-k distributions. *Journal of Statistical Planning and Interference*, 65:45-66.

Hoaglin, D. C. (1983): *Summarizing shape numerically: The $g-and-h$ distributions*, in: Hoaglin, D. C. and F. Mosteller and J. W. Tukey (eds.): Data analysis for tables, trends, and shapes, Wiley, New York.

Klein, I. and M. Fischer (2003): Kurtosis transformation and kurtosis ordering. *Diskussion paper 53*, University of Erlangen-Nürnberg, (submitted).

MacGillivray, H. L. and K. P. Belanda (1988): The relationship between skewness and kurtosis. *Australian Journal of Statistics*, 30(3):319-337.

Martinez, J. and B. Iglewicz (1984): Some properties of the Tukey g and h family of distributions. *Communication in Statistics (Theory and Methods)*, 13:353-369.

McDonald, J. B. (1991): Parametric Models for Partial Adaptive Estimation with Skewed and Leptokurtic Residuals. *Econometric Letters*, 37:273-288.

Mills, T. C. (1995): Modelling skewness and kurtosis in the London Stock exchange FT-SE index return distribution. *The Statistician*, 44(3):323-332.

Morgenthaler, S. and J. W. Tukey (2000): Fitting Quantiles: Doubling, HR, HQ, and HHH Distributions. *Journal of Computational and Graphical Statistics*, 9(1):180-195, 2000.

Morris, C. N. (1982): Natural Exponential Families with Quadratic Variance Functions. *The Annals of Statistics*, **10**(1):65-80.

Oja, H. (1981): On location, scale, skewness and kurtosis of univariate distributions. *Scandinavian Journal of Statistics*, 18:154-168.

Prause, K. (1999): *The Generalized Hyperbolic Model: Estimation, Financial Derivatives and Risk Measures*. PhD Thesis, Universität Freiburg, Freiburg.

Theodossiou, P. (1998): Financial data and the Skewed Generalized t distribution. *Mathematical Science*, 44(12):1650-1660.

Tukey, J. W. (1960): *The practical relationship between the common transformations of counts of amounts*. Princeton University Statistical Techniques Research Group, Technical Report No. 36.

Tukey, J. W. (1977): *Exploratory data analysis*. Addison-Wesley, Reading, M.A.