Tailoring copula-based multivariate generalized hyperbolic secant distributions to financial return data: An empirical investigation

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Abstract

On of the crucial questions in risk management is how to aggregate individual risk into overall portfolio risk.

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1 Introduction

2 Multivariate distributions: The copula approach

There are several possibilites to construct two-dimensional distribution families proposed in the literature, one of them being the so-called copula approach, where "copulas" play the central part. The axiomatic definition of such a (two dimensional) copula is as follows:

Definition 2.1 (2-Copula) *A two-dimensional copula (or 2-copula) is a function* $C : [0,1] \times [0,1] \rightarrow [0,1]$ *which satisfies the following properties:*

1. *C is 2-increasing, i.e. for $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$ holds:*

$$C(v_1, v_2) - C(v_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \geq 0.$$

2. *For all $u, v \in [0,1]$: $C(u,0) = C(0,v) = 0$ and $C(u,1) = C(1,v) = u$.*

Theorem 2.1 (Sklar) *Let F_1 and F_2 be two univariate distribution functions. Then $C(F_1(x_1), F_2(x_2))$ defines a bivariate probability distribution with margins F_1 und F_2. On the other hand, let F be a 2-dimensional distribution function with margins F_1 and F_2. Then F has the copula representation $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$. The copula C is unique if the margins are continuous.*

Following this fundamental theorem of Sklar (1957), the procedure to construct a multivariate distribution is fairly simple at first sight:

1. Choose a suitable copula C that can rebuild the dependence structure between financial returns.

2. Choose two suitable marginal distributions F_X and F_Y which rebuild the distributional stylized facts of log-returns.

3. The bivariate distribution function can then be obtained by means of

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \tag{1}$$

   or, in terms of a bivariate density function, by means of

$$f_{X,Y}(x,y) = c(F_X(x), F_Y(y)) \cdot f_X(x) \cdot f_Y(y) \text{ with } c(x,y) = \frac{\partial^2 C(x,y)}{\partial x \partial y} \tag{2}$$

It is obvious that the main obstacle is not to construct the bivariate distribution (assuming that the margins and the copula are given) but to select the margins and – above all – the copulas knowing that there are dozens of proposals for a copula (see, for example, Joe (1997)) and several flexible multi-parametric distribution families. For that reason we postulate certain requirements on both the margins and the copulas which will be discussed in the next two sections.

3 The choice of the marginal distribution

A marginal distribution should essentially fulfil two requirements: Firstly and most important, the underlying distribution model is proclaimed to be flexible with respect to different combination of skewness and kurtosis. Secondly, we also proclaim certain "numerical" properties. In particular – according to equation (2) – a cumulative distribution function in closed form will avoid numerical integration methods and save computational time. In the literature, several flexible distribution families have been proposed up to now. All of them more or less seem to satisfy the first requirement. Take for example the generalized hyperbolic family (see Eberlein and Keller (1995) or Prause (1999)), the generalized logistic family (see McDonald (1992) or Fischer (2002)), the α-stable family (see Mittnik, Paolella and Rachev (1997)) or families of generalized t-distributions (see Theodossiou (1998) or Grottke (2001)). However, neither the generalized hyperbolic distribution nor the family of generalized logistic distribution, neither the α-stable family nor the generalized t-distributions provide a closed form for the cumulative distribution function. For that reason we put emphasize on a distribution to which not much attention was paid in the context of financial return data: The family of skewed generalized secant distribution which has its origin in the hyperbolic secant distribution which was studied first by Baten(1934) and Talacko (1956) and has density

$$f_{HS}(x) = \frac{1}{2\cosh(\pi x/2)}$$

Since 1956, two generalization have been proposed which allow for a more flexible form concerning skewness and leptokurtosis: The first proposal was introduced by Morris (1982) in the context of natural exponential families (NEF) with quadratic variance function (i.e. the variance is a quadratic function of the mean). In this class with six members, one is generated by the hyperbolic secant distribution, the so-called NEF-GHS distribution with pdf

$$f(x;\lambda,\theta) = \frac{2^{\lambda-2}}{\pi\,\Gamma(\lambda)} \cdot \left|\Gamma\left(\frac{\lambda+ix}{2}\right)\right|^2 \cdot \exp\{\theta x + \lambda\log(\cos(\theta))\}$$

for $\lambda > 0$ and $|\theta| < \pi/2$. The NEF-GHS distribution allows for skewness and arbitrarily high excess kurtosis. Morris showed that this class is again reproductive, infinitely divisible with existing moment-generating function and existing moments. However, the corresponding cumulative distribution function doesn't admit a closed form.

Recently, Vaughan (2002) proposes a family of symmetric distributions — which he called *generalized secant hyperbolic (GSH) distribution* — with kurtosis ranging from 1.8 to infinity which includes both the hyperbolic secant and the logistic distribution and closely approximates the Student t-distribution with corresponding kurtosis. The probability density function of a GSH variate is given by

$$f_{GSH}(x;t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t)\exp(c_2(t)x) + 1}, \qquad x \in \mathbb{R} \tag{3}$$

with

$$\begin{array}{lllll} a(t) = \cos(t), & c_2(t) = \sqrt{\frac{\pi^2 - t^2}{3}} & c_1(t) = \frac{\sin(t)}{t} \cdot c_2(t), & \text{for } -\pi < t \leq 0, \\ a(t) = \cosh(t), & c_2(t) = \sqrt{\frac{\pi^2 + t^2}{3}} & c_1(t) = \frac{\sinh(t)}{t} \cdot c_2(t), & \text{for } t > 0 \end{array}.$$

The density from (3) is chosen so that X has zero mean and unit variance. The GSH distribution includes the logistic distribution ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as limiting case for $t \to \infty$. Vaughan derives the cumulative distribution function, depending on the parameter t, as

$$F_{GSH}(x;t) = \begin{cases} 1 + \frac{1}{t} \operatorname{arccot}\left(-\frac{\exp(c_2(t)x) + \cos(t)}{\sin(t)}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\exp(\pi x/\sqrt{3})}{1 + \exp(\pi x/\sqrt{3})} & \text{for } t = 0, \\ 1 - \frac{1}{t} \operatorname{arccoth}\left(\frac{\exp(c_2(t)x) + \cosh(t)}{\sinh(t)}\right) & \text{for } t > 0. \end{cases} \tag{4}$$

the inverse distribution function

$$F_{GSH}^{-1}(u;t) = \begin{cases} \frac{1}{c_2(t)} \ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \text{für } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi} \ln\left(\frac{u}{1-u}\right) & \text{für } t = 0, \\ \frac{1}{c_2(t)} \ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \text{für } t > 0. \end{cases}$$

In addition, the moment-generating function and all moments exist, and the cumulative distribution is given in closed form. Unfortunately, this family does not allow for skewness. For this purpose, we introduce a skewness parameter by means of splitting the scale parameter according to Fernandez, Osiewalski and Steel (1995). In particular, this transformation preserves the closed form for the density, given by

$$f_{SGSH}(x;t,\gamma) = \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ f_{GSH}(x/\gamma) \cdot \mathbf{I}^-(x) + f_{GSH}(\gamma x) \cdot \mathbf{I}^+(x) \right\} \tag{5}$$

$$= \frac{2c_1}{\gamma + \frac{1}{\gamma}} \cdot \left(\frac{\exp(c_2 x/\gamma) \cdot \mathbf{I}^-(x)}{\exp(2c_2 x/\gamma) + 2a\exp(c_2 x/\gamma) + 1} + \frac{\exp(c_2 \gamma x) \cdot \mathbf{I}^+(x)}{\exp(2c_2 \gamma x) + 2a\exp(c_2 \gamma x) + 1} \right),$$

where is symmetric for $\gamma = 1$, skewed to the right for $\gamma > 1$ and skewed to the left for $0 < \gamma < 1$. The corresponding distribution will be termed as *skewed generalized secant hyperbolic distribution* in the sequel. The corresponding cumulative distribution function is given by

$$F_{SGSH}(x;t,\gamma) = \frac{2\gamma^2}{\gamma^2 + 1} \cdot \left(F_{GSH}\left(x/\gamma\right) \cdot \mathbf{I}^-(x) + \left(\frac{\gamma^2 - 1 + 2F_{GSH}(\gamma x)}{2\gamma^2} \right) \cdot \mathbf{I}^+(x) \right),$$

More details can be found in Fischer and Vaughan (2002), where the inverse cumulative distribution function and the moments are calculated. It is also shown that this family provides an excellent fit to financial return data.

4 The choice of the copula

Much more less explored than the choice of the marginal distribution is the selection problem of an "optimal" copula. Due to Durrleman et al. (2000) we assume that we have a finite subset of copulas $\mathcal{C}_0 \subset \mathcal{C}$. As we do not restrict our research to elliptical copulas, we also use certain distance measures between the empirical copula and some parametric copulas: Let $\{x_n^1, x_n^2\}_{n=1,\ldots,N}$ be a bivariate sample of size N. The *empirical copula* dates back to Deheuvel (1979) and is defined by

$$C_{emp}\left(i^*, j^*\right) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\{x_n^1 \leq x_{(i)}^1, x_n^2 \leq x_{(j)}^2\}}, \tag{6}$$

where $i^* = i/N$ and $j^* = j/N$. To judge the goodness-of-fit we calculate several distance measures between the empirical copula C_{emp} and the (estimated) parametric copula C_{para}. Putting emphasize on the deviations in the tails, *Anderson-Darling statistic* is calculated

$$\mathcal{AD}_0 = \sqrt{N} \cdot \max_{1 \leq i,j \leq N} \frac{|C_{emp}(i^*, j^*) - C_{para}(i^*, j^*)|}{\sqrt{(C_{para}(i^*, j^*))(1 - C_{para}(i^*, j^*))}} \tag{7}$$

Taking also the second and third largest value into account, we also calculate $\mathcal{AD}_1$ and $\mathcal{AD}_2$. As a global goodness-of-fit statistic which has still focus on the tails we also use the *integrated Anderson-Darling statistic* given by

$$\mathcal{IAD} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{(C_{emp}(i^*, j^*) - C_{para}(i^*, j^*))^2}{C_{para}(i^*, j^*)(1 - C_{para}(i^*, j^*))}. \tag{8}$$

In addition, we also calculate a distance based on the discrete $\mathcal{L}^2$-norm by

$$\mathcal{L}_2 = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} (C_{emp}(i^*, j^*) - C_{para}(i^*, j^*))^2}. \tag{9}$$

We will essentially focus on three classes (which are not disjunct) of copulas
Elliptical copulas in general cannot be expressed in closed form. Moreover, only symmetric tail dependence can be captured. Examples are the Gaussian copula,

$$C_G = C_G(u, v, \rho) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$$

the t-copula

$$C_t = C_t(u, v; \rho, \nu) = T_{\nu,\rho}(t_\nu^{-1}(u), t_\nu^{-1}(v)).$$

Up to There are two classes (which are not disjunct!) of copulas

1. *Elliptical copulas* associated to elliptical distributions are widely used in statistics and econometrics, especially in finance. Applications and limitations are discussed by Frahm, Junker and Szimayer (2002). Popular in finance are the multivariate Gaussian, the multivariate Student t and the sub-Gaussian α-stable distribution.

2. *Archimedean copulas* with generator φ which are defined by

$$C(u,v) = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right), \qquad (10)$$

where φ is a continuous, convex and strictly decreasing function from $[0,1]$ to $[0,\infty]$ and $\varphi(1) = 0$, and φ^{-1} denotes its pseudo-inverse. Popular examples are the Frank copula with $\varphi(u) = -\ln\left[(e^{-\theta t} - 1)/(e^{-\theta} - 1)\right]$, the Clayton copula with $\varphi(t) = (t^{-\theta} - 1)/\theta$ or the Gumbel copula with $\varphi(t) = -\ln(t)^{\theta}$. If C is absolutely continuous, its density is given by

$$c(u,v) = -\frac{\varphi''(C(u,v))\varphi'(u)\varphi'(v)}{[\varphi(C'(u,v))]^3}. \qquad (11)$$

Furthermore, for Archimedean copulas, tail dependence can be expressed in terms of the generators φ (see, for example, Embrechts, Lindskog and McNeil (2002)). Members of this class can have both symmetric and asymmetric tail dependence. It is easy constructing a new Archimedean copula by constructing a new generator via $\overline{\varphi} = g \circ \varphi$, where $g : [0,1] \to [0,1]$ is strictly monotone increasing, concave with $g(1) = 1$. In order to introduce tail dependence, Junker and May (2002) chose $g(x) = x^{\delta}$, $\delta \geq 1$. The so-called transformed Frank copula is given by

$$C(u,v) \;=\; -\frac{1}{\theta} \cdot \ln\left[1 + (e^{-\theta} - 1)\exp\left(-\{\varphi(u;\theta)^{\delta} + \varphi(v;\theta)^{\delta}\}^{1/\delta}\right)\right]$$

with density given in (11), where

$$\overline{\varphi}'(t;\theta,\delta) = \frac{\delta\theta}{1 - e^{\theta t}} \cdot \varphi(t,\theta)^{\delta-1}$$

$$\overline{\varphi}''(t;\theta,\delta) = \frac{\delta\theta^2 e^{-\theta t}}{(e^{-\theta t} - 1)^2} \cdot \Big(e^{-\theta t}(d-1) + \varphi(t,\theta)\Big) \cdot \varphi(t,\theta)^{\delta-2}$$

Clearly, for $\delta = 1$ we have the Frank copula. Junker and May (2002) showed that C_{TF} has no lower tail dependence but is upper tails dependent with tail dependence parameter $\lambda_U = 2 - 2^{1/\delta}$.

Possibilities to construct new copulas, as for example

1. Rotation of copulas by means of

$$\overline{C}(u,v) = u + v - 1 + C(u,v)$$

2. It can be easily shown that any convex combination of two (or more generally of n) copulas C_1 and C_2

$$C(u,v) = \lambda C_1(u,v) + (1-\lambda)C_2(u,v) \text{ for } \lambda \in [0,1]$$

is again a copula. In order to allow both for lower and for upper tail dependence, Junker and May (2002) propose to convex-combine a Cook-Johnson (Clayton) copula and a survival copula of the same family:

$$\begin{aligned} C_{cCJ} &= \alpha \cdot \overline{C}_{JC}(u,v;\delta_1) + (1-\alpha) \cdot C_{JC}(u,v;\delta_2) \\ &= \alpha \cdot (u+v-1+C_{JC}(1-u,1-v;\delta_1)) + (1-\alpha) \cdot C_{JC}(u,v;\delta_2), \end{aligned} \quad (12)$$

where $\alpha \in [0,1]$. The convex-combined Cook-Johnson copula C_{cCJ} is lower tail dependent with parameter $\lambda_L = \alpha 2^{-1/\delta_1}$ and upper tail dependent with parameter $\lambda_U = (1-\alpha)2^{-1/\delta_2}$. In order to fulfil certain requirements (i.e. asymmetric tail dependence, whole range for lower tail dependence, any degree of Spearman's rank correlation coefficient and nesting the Gaussian copula), Fortin and Kuzmics (2002) convex-combined a t-copula/Gaussian copula with a Clayton copula, a rotated Gumbel and a rotated Joe copula.

3. Transforming copulas: Durrleman, Nikeghbali and Roncalli (2000) showed that

$$C_\gamma(u,v) = \gamma^{-1}(C(\gamma(u),\gamma(v)))$$

is a copula if $\gamma : [0,1] \to [0,1]$ is a concave bijection with $\gamma(0) = 0$ and $\gamma(1) = 1$, which is C^1-diffeomorphism from $(0,1)$ onto $(0,1)$ and is twice differentiable on $(0,1)$. Examples are

$$\gamma(x) = x^{1/\beta}, \quad \beta \geq 1, \quad \gamma(x) = \sin(\pi x/2), \quad \gamma(x) = \frac{\beta_1 x + \beta_2 x}{\beta_1 x + \beta_2}, \quad \beta_1, \beta_2 > 0.$$

4. Construction via Laplace transforms: Let F be the cumulative distribution function of a positive random variable X_F. Then the Laplace transform ψ of F is defined as

$$\psi(s) = \int_0^\infty e^{-sx} dF(x)$$

For every copula K we can derive another copula C via

$$C(u,v) = \psi\Big[-\log K(e^{-p_1\psi^{-1}(u)}, -p_2\psi^{-1}(v)) + \nu_1 p_1 \psi^{-1}(u) + \nu_2 p_2 \psi^{-1}(v)\Big]$$

with $\nu_i \geq 0$ and $p_i = \frac{1}{\nu_i + 1}$. In particular, setting $\nu_1 = \nu_2 = 0$ implying $p_1 = p_2 = 1$,

$$C(u,v) = \psi\Big[-\log K(e^{-\psi^{-1}(u)}, e^{-\psi^{-1}(v)})\Big] \quad (13)$$

Note, that for the independence copula $K(u,v) = uv$ equation (13) reduces to (10). Several example can be found in Joe (2000): Combining the Gumbel copula with Laplace transform LTB [LTC] we end up with the two-parameter copula family $BB1$ [$BB7$ or Joe-Clayton], Clayton copula in combination with LTA generates copula family $BB3$, Galambos copula with LTB generate copula family $BB4$, the independence copula with the two-parameter family LTL results in copula family $BB10$. Furthermore, let K denote a Frank copula and ψ the Laplace transform LTC. With $\breve{x} = 1 - e^{-\delta\left(1-\bar{x}^{\theta}\right)}$, the corresponding copula (we call it $BBMF$) is

$$C = C(u,v;\theta,\delta) = 1 - \left(1 + \ln\left(1 - \frac{\breve{u}\breve{v}}{1-\exp(-\delta)}\right)\frac{1}{\delta}\right)^{\frac{1}{\theta}}.$$

It can be shown that the corresponding copula density is given by

$$c(u,v;\theta,\delta) = \left(\frac{1-C}{\bar{u}\bar{v}}\right)^{1-\theta} \frac{(1-\breve{u})(1-\breve{v})}{\left(1-e^{-\delta}-\breve{u}\breve{v}\right)^2} \left\{\frac{\breve{u}\breve{v}(\theta-1)}{(1-C)^{\theta}} + \delta\theta\left(1-e^{-\delta}\right)\right\}$$

Elliptical copulas:

$$c(u,v;\rho) = \frac{1}{\sqrt{det(\Sigma)}} \cdot \exp(-0.5\zeta'(\Sigma^{-1} - \mathbf{I}_2)\zeta)$$

$$c(u,v;\rho,\nu) = \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{\Gamma((\nu+2)/2)\Gamma(\nu/2)}{(\Gamma((\nu+1)/2))^2} \cdot \frac{\left[(1+\frac{\Phi^{-1}(u)^2}{\nu})\left(1+\frac{\Phi^{-1}(v)^2}{\nu}\right)\right]^{\frac{\nu+1}{2}}}{(1+\frac{\zeta'\Sigma^{-1}\zeta}{\nu})^{\frac{\nu+2}{2}}}$$

Independence or product copula

$$C_I = C_\Pi(u,v) = u \cdot v$$

The minimum copula

$$C_L = C_L(u,v) = \max(u+v-1,0)$$

The maximum copula

$$C_U = C_U(u,v) = \min(u,v)$$

Tail dependence: If C is a bivariate copula such that

$$\lim_{u \to 1} \frac{\overline{C}(u,u)}{1-u} = \lim_{u \to 1} P(F_X(x) > 1-u | F_Y(y) > 1-u) = \lambda_U$$

exists, then C is said to be upper tail dependent if $\lambda_U \in (0,1]$ and no upper tail dependent if $\lambda_U = 0$. Similarly, if

$$\lim_{u \to 0} \frac{C(u,u)}{u} = \lambda_L$$

exists, C has lower tail dependence if $\lambda_L \in (0,1]$ and no lower tail dependence if $\lambda_L = 0$.

Tail dependence for a convex combination $C(u,v) = \sum_i \lambda_i C_i(u,v)$ mit $\sum_i \lambda_i = 1$. Tail dependence of the rotated copula: The lower (upper) tail dependence of C^r is the same as the upper (lower) tail dependence of C.

Table 1: Archimedean copulas

Name	$\varphi(u)$	$\varphi(u)^{-1}$
$B0$	$-\ln(u)$	$\exp(-u)$
$B3$	$-\log\left(\frac{\exp(-\alpha u)-1}{\exp(-\alpha)-1}\right)$	
$B4$	$u^{-\alpha} - 1$	$(u+1)^{-1/\alpha}$
$B6$	$(-\ln(u))^{\alpha}$	$\exp(-u^{1/\alpha})$
$B5$		

5 Application to bivariate financial return data

Throughout this work we focus on the bivariate analysis

Table 2: Parameterized copulas

name	Abr.	formula	parameter
Minimum	BL	$\max(u+v-1,0)$	–
Maximum	BU	$\min(u,v)$	–
Independence	$B0$	$u \cdot v$	–
		Bivariate one-parameter families	
Normal	$B1$	$\Phi_\rho(\Phi^{-1}(u),\Phi^{-1}(v))$	$\lvert\rho\rvert \leq 1$
Plackett	$B2$	$\frac{1}{2\eta}(1+\eta(u+v)-\sqrt{(1+\eta(u+v))^2-4\delta\eta uv})$	$\delta \geq 0$
Frank	$B3$	$-\frac{1}{\delta}\log\left(1-\frac{(1-e^{-\delta u})(1-e^{-\delta v})}{1-e^{-\delta}}\right)$	$\delta \geq 0$
Clayton	$B4$	$(u^{-\delta}+v^{-\delta}-1)^{-\frac{1}{\delta}}$	$\delta \geq 0$
Joe	$B5$	$1-(\bar{u}^\delta+\bar{v}^\delta-\bar{u}^\delta\bar{v}^\delta)^{1/\delta}$	$\delta \geq 1$
Gumbel	$B6$	$\exp(-(\widetilde{u}^\delta+\widetilde{v}^\delta)^{1/\delta})$	$\delta \geq 1$
Galambos	$B7$	$uv\exp((\tilde{u}^{-\delta}+\tilde{v}^{-\delta})^{-1/\delta})$	$\delta \geq 0$
Hüsler-Reiss	$B8$	$\exp\left(-\tilde{u}\Phi\left[\frac{1}{\delta}+\frac{\delta}{2}\ln(\frac{\tilde{u}}{\tilde{v}})\right]-\tilde{v}\Phi\left[\frac{1}{\delta}+\frac{\delta}{2}\ln(\frac{\tilde{v}}{\tilde{u}})\right]\right)$	$\delta \geq 0$
Rotated Clayton	$B4'$	$u+v-1+(\bar{u}^{-\delta}+\bar{v}^{-\delta}-1)^{-\frac{1}{\delta}}$	$\delta \geq 0$
Rotated Joe	$B5'$	$u+v-(u^\delta+v^\delta-u^\delta v^\delta)^{1/\delta}$	$\delta \geq 1$
Rotated Gumbel	$B6'$	$u+v-1+\exp(-(\widetilde{\bar{u}}^\delta+\widetilde{\bar{v}}^\delta)^{1/\delta})$	$\delta \geq 1$
Ali-Mikhail-Haq	$B12$	$(uv)/(1-\delta\cdot\bar{u}\bar{v})$	$\lvert\delta\rvert \leq 1$
		Bivariate two-parameter families	
	$BB1$	$\left(1+[\hat{u}^\delta+\hat{v}^\delta]^{1/\delta}\right)^{-1/\theta}$	$\theta \geq 1, \delta > 0$
	$BB2$	$\exp\left(-\left[\frac{1}{\delta}\log(\exp(\delta\tilde{u}^\theta)+\exp(\delta\tilde{v}^\theta)-1)\right]^{1/\theta}\right)$	$\theta > 0, \delta \geq 1$
	$BB4$	$\left(\hat{u}+\hat{v}+1-\left[\hat{u}^{-\delta}+\hat{v}^{-\delta}\right]^{-\frac{1}{\delta}}\right)^{-\frac{1}{\theta}}$	$\theta \geq 0, \delta > 0$
Joe-Clayton	$BB7$	$1-\left(1-\left[(1-\bar{u}^\kappa)^{-\delta}+(1-\bar{v}^\kappa)^{-\delta}-1\right]^{-\frac{1}{\delta}}\right)^{\frac{1}{\kappa}}$	$\kappa \geq 1, \delta > 0$
Transf. Frank	$BB10$	$-\theta^{-1}\cdot\ln\left[1+\frac{\exp\left(-\{\varphi(u;\theta)^\delta+\varphi(v;\theta)^\delta\}^{1/\delta}\right)}{(e^{-\theta}-1)^{-1}}\right]$	$\theta > 0, \delta \geq 0$
t	$BB11$	$T_{\nu,\rho}(t_\nu^{-1}(u),t_\nu^{-1}(v))$	$\lvert\rho\rvert \leq 1, \nu > 0$
MF	$BBMF$	$1-\left(1+\ln\left(1-\frac{\breve{u}\breve{v}}{1-\exp(-\delta)}\right)\frac{1}{\delta}\right)^{\frac{1}{\theta}}$	$\theta > 1, \delta \geq 1$
		Bivariate three-parameter families	
CC Clayton	$BC1$	$\alpha C_{B4}(u,v;\delta_1)+(1-\alpha)\overline{C_{B4}}(u,v;\delta_2)$	$\delta_i > 0, \alpha \in [0,1]$
CC Joe	$BC2$	$\alpha C_{B5}(u,v;\delta_1)+(1-\alpha)\overline{C_{B5}}(u,v;\delta_2)$	$\delta_i \geq 1, \alpha \in [0,1]$
CC Normal-Clayton	$BC3$	$\alpha C_{B4}(u,v;\delta)+(1-\alpha)C_{B1}(u,v;\rho)$	$\delta > 0, \lvert\rho\rvert \leq 1, \alpha \in [0,1]$

Note, that $\bar{u}=1-u$, $\hat{u}=u^{-\theta}-1$, $\tilde{u}=-\log(u)$, $\bar{C}$ Survival copula.

Table 3: Tail dependence and limiting cases

name	τ^L	τ^U	Dependence
Minimum			C_L
Maximum			C_U
Independence			C_I
Bivariate one-parameter families			
Normal $(B1)$	0	0	$C_L\,(\rho=-1), C_I\,(\rho=0), C_U\,(\rho=1)$
Plackett $(B2)$			$C_L\,(\delta\to 0), C_I\,(\delta\to 1), C_U\,(\delta\to\infty)$
Frank $(B3)$	0	0	$C_I\,(\delta\to 0), C_U\,(\delta\to\infty)$
Clayton $(B4)$	$2^{-1/\delta}$	0	$C_I\,(\delta\to 0), C_U\,(\delta\to\infty)$
Joe $(B5)$	0	$2-2^{1/\delta}$	$C_I\,(\delta=1), C_U\,(\delta\to\infty)$
Gumbel $(B6)$	0	$2-2^{1/\delta}$	$C_I\,(\delta=1), C_U\,(\delta\to\infty)$
Galambos $(B7)$			$C_I\,(\delta\to 0), C_U\,(\delta\to\infty)$
Hüsler-Reiss $(B8)$			$C_I\,(\delta\to 0), C_U\,(\delta\to\infty)$
Rotated Clayton $(B4')$	0	$2^{-1/\delta}$	
Rotated Joe $(B5')$	$2-2^{1/\delta}$	0	
Rotated Gumbel $(B6)$	$2-2^{1/\delta}$	0	
Ali-Mikhail-Haq			
Bivariate two-parameter families			
$BB1$	$2^{-1/(\delta\theta)}$	$2-2^{1/\delta}$	$B4\,(\theta=1),\ B6\,(\delta\to 0)$
$BB3$	1	$2-2^{1/\theta}$	$B4\,(\theta=1),\ B6\,(\delta\to 0)$
$BB4$	$(2-2^{1/\delta})^{-1/\theta}$	$2^{-1/\delta}$	$B7\,(\theta\to 0),\ B4\,(\delta\to 0)$
$BB7$	$2^{-1/\delta}$	$2-2^{-1/\kappa}$	$B4\,(\kappa=1),\ B5\,(\delta\to 0)$
$BB10$	0	$2-2^{1/\delta}$	$B3\,(\delta=1)$
$BB11$	$2\left(1-t_{\nu+1}\left(\frac{\sqrt{\nu+1}\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\right)$		$C_L\,(\rho=-1), C_I\,(\rho=0), C_U\,(\rho=1)$
Bivariate three-parameter families			
CC Clayton	$\alpha 2^{-1/\delta_1}$	$(1-\alpha)2^{-1/\delta_2}$	$B4\,(\alpha=1), B4'\,(\alpha=0)$
CC Joe	$\alpha(2-2^{1/\delta})$	$(1-\alpha)(2-2^{1/\delta})$	$B5\,(\alpha=1), B5'\,(\alpha=0)$
CC Normal-Clayton	$\alpha 2^{-1/\delta}$	0	$B4\,(\alpha=1), B1\,(\alpha=0)$

	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\mathcal{IAD}$	$\mathcal{KS}$	$\mathcal{LL}$	$\mathcal{AIC}$	χ^2
	Independence copula							
Norm	0.619	0.607	0.604	51.495	85.129	-3699.439	7408.929	
SH	0.619	0.607	0.604	51.495	85.129	-3660.336	7330.722	
t	0.619	0.607	0.604	51.495	85.129	-3652.784	7319.663	
GSH	0.619	0.607	0.604	51.495	85.129	-3653.702	7325.556	
SGSH	0.619	0.607	0.604	51.495	85.129	-3643.008	7304.169	
	Gaussian copula							
Norm	0.262	0.262	0.262	0.307	6.087	-3290.909	6593.889	
SH	0.262	0.262	0.262	0.254	4.839	-3231.147	6474.364	
t	0.262	0.262	0.262	0.275	5.476	-3225.345	6466.894	
GSH	0.262	0.262	0.262	0.277	5.435	-3227.928	6476.042	
SGSH	0.262	0.262	0.262	0.273	5.450	-3216.788	6453.788	
	t-copula							
Norm	0.064	0.061	0.058	0.096	4.500	-3287.724	6589.542	
SH	0.062	0.059	0.056	0.056	4.186	-3228.954	6472.003	
t								
GSH								
SGSH								
	Frank copula							
Norm	0.262	0.220	0.216	0.416	8.679	-3274.776	6561.622	
SH	0.262	0.232	0.230	0.289	5.742	-3254.022	6520.114	
t	0.262	0.229	0.227	0.301	6.230	-3240.971	6498.063	
GSH	0.262	0.228	0.226	0.305	6.378	-3241.456	6503.098	
SGSH	0.262	0.229	0.226	0.303	6.301	-3232.565	6485.316	
	Plackett copula							
Norm	0.198	0.161	0.157	0.167	5.124	-3281.934	6575.938	
SH	0.211	0.171	0.169	0.214	5.749	-3252.308	6516.687	
t	0.210	0.171	0.168	0.210	5.677	-3243.102	6502.326	
GSH	0.209	0.170	0.168	0.205	5.601	-3243.676	6507.537	
SGSH	0.210	0.171	0.168	0.209	5.664	-3235.040	6490.265	

Table 4: Goodness-of-fit for different copulas I

	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\mathcal{IAD}$	$\mathcal{KS}$	$\mathcal{LL}$	$\mathcal{AIC}$	χ^2
	Ali-Mikhail-Haq copula							
Norm	0.147	0.146	0.145	4.221	33.410	-3381.853	6775.777	
SH	0.146	0.145	0.144	4.115	33.028	-3326.394	6664.858	
t	0.146	0.145	0.144	4.115	33.029	-3321.289	6658.700	
GSH	0.146	0.145	0.144	4.138	33.114	-3322.734	6665.653	
SGSH	0.146	0.145	0.145	4.177	33.253	-3315.525	6651.236	
	Clayton copula							
Norm	0.103	0.102	0.102	1.681	20.278	-3374.627	6761.324	
SH	0.101	0.101	0.100	1.628	19.807	-3292.300	6596.670	
t	0.102	0.101	0.101	1.644	19.953	-3289.384	6594.889	
GSH	0.101	0.101	0.101	1.641	19.921	-3291.867	6603.919	
SGSH	0.097	0.097	0.097	1.548	19.036	-3290.960	6602.107	
	Rotated Clayton copula							
Norm	0.437	0.430	0.426	9.117	33.560	-3469.762	6951.595	
SH	0.407	0.401	0.396	5.814	25.412	-3346.311	6704.692	
t	0.400	0.394	0.389	5.163	23.545	-3332.888	6681.898	
GSH	0.408	0.402	0.398	5.917	25.697	-3344.691	6709.567	
SGSH	0.376	0.371	0.366	3.400	17.871	-3309.397	6638.980	
	Gumbel copula							
Norm	0.159	0.154	0.150	0.869	11.387	-3377.305	6766.680	
SH	0.154	0.150	0.146	0.730	10.317	-3280.451	6572.972	
t	0.153	0.149	0.145	0.693	10.011	-3273.972	6564.066	
GSH	0.155	0.150	0.146	0.736	10.365	-3280.408	6581.001	
SGSH	0.137	0.133	0.129	0.351	6.767	-3244.594	6509.375	
	Joe-Clayton copula							
Norm	0.092	0.091	0.090	1.124	16.924	-3347.234	6708.563	
SH	0.080	0.079	0.078	0.579	11.583	-3247.610	6509.314	
t	0.079	0.079	0.078	0.518	10.917	-3241.816	6500.384	
GSH	0.080	0.079	0.078	0.591	11.727	-3247.610	6517.443	
SGSH	0.074	0.073	0.072	0.486	10.854	-3241.532	6505.287	

Table 5: Goodness-of-fit for different copulas II

	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\mathcal{IAD}$	$\mathcal{KS}$	$\mathcal{LL}$	$\mathcal{AIC}$	χ^2
	Joe copula							
Norm	0.441	0.435	0.430	8.979	32.426	-3485.862	6983.794	
SH	0.420	0.414	0.409	6.597	26.779	-3362.337	6736.745	
t	0.413	0.408	0.403	5.938	25.047	-3348.603	6713.328	
GSH	0.422	0.416	0.411	6.755	27.183	-3360.708	6741.603	
SGSH	0.386	0.382	0.376	3.854	18.937	-3321.045	6662.275	
	Hüsler-Reiss copula							
Norm	0.216	0.210	0.205	3.626	24.455	-3463.404	6938.879	
SH	0.174	0.169	0.165	1.317	14.330	-3300.558	6613.186	
t	0.164	0.159	0.155	0.934	11.847	-3283.977	6584.075	
GSH	0.174	0.170	0.165	1.329	14.403	-3300.306	6620.797	
SGSH	0.146	0.142	0.137	0.455	7.845	-3256.091	6532.368	
	Galambos copula							
Norm	0.161	0.157	0.152	0.899	11.602	-3393.960	6799.990	
SH	0.158	0.153	0.149	0.780	10.707	-3283.701	6579.474	
t	0.155	0.151	0.147	0.718	10.207	-3275.393	6566.907	
GSH	0.158	0.153	0.149	0.789	10.776	-3283.606	6587.397	
SGSH	0.138	0.134	0.130	0.350	6.748	-3245.811	6511.808	
	BB3 copula							
Norm	0.096	0.095	0.094	1.206	17.381	-3357.775	6729.644	
SH	0.073	0.073	0.072	0.434	10.078	-3247.201	6508.497	
t	0.076	0.076	0.075	0.399	9.477	-3238.928	6496.007	
GSH	0.073	0.072	0.072	0.499	10.945	-3247.855	6517.934	
SGSH	0.057	0.055	0.054	0.302	8.001	-3234.625	6491.474	
	BB1 copula							
Norm	0.071	0.070	0.069	0.397	9.555	-3320.232	6654.558	
SH	0.072	0.071	0.070	0.290	7.589	-3233.869	6481.833	
t	0.070	0.070	0.069	0.285	7.605	-3229.056	6476.265	
GSH	0.071	0.071	0.070	0.298	7.775	-3233.525	6489.274	
SGSH	0.054	0.053	0.053	0.186	6.470	-3225.909	6474.041	

Table 6: Goodness-of-fit for different copulas III

	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$	$\mathcal{IAD}$	$\mathcal{KS}$	$\mathcal{LL}$	$\mathcal{AIC}$	χ^2
	Transformed Frank copula							
Norm	0.219	0.196	0.195	0.325	7.588	-3267.640	6549.375	
SH	0.174	0.169	0.169	0.149	3.881	-3241.531	6497.157	
t	0.185	0.182	0.181	0.175	4.430	-3231.596	6481.344	
GSH	0.190	0.189	0.185	0.187	4.683	-3232.691	6487.604	
SGSH	0.172	0.168	0.167	0.151	4.117	-3221.541	6465.306	
	Convex-combined Clayton copula							
Norm	0.063	0.063	0.062	0.258	6.856	-3294.796	6605.713	
SH	0.049	0.048	0.047	0.186	6.645	-3241.571	6499.263	
t	0.049	0.048	0.047	0.191	6.756	-3235.988	6492.161	
GSH	0.048	0.048	0.048	0.186	6.634	-3240.200	6504.664	
SGSH	0.047	0.046	0.045	0.181	6.345	-3233.567	6495.295	
	BBMF copula							
Norm	0.262	0.218	0.214	0.391	8.334	-3269.644	6553.383	
SH	0.262	0.221	0.218	0.239	5.389	-3246.859	6507.812	
t	0.262	0.222	0.219	0.264	5.872	-3235.361	6488.874	
GSH	0.262	0.223	0.220	0.274	6.059	-3236.273	6494.770	
SGSH	0.262	0.222	0.219	0.265	5.906	-3228.640	6479.503	
	Convex-combined Joe copula							
Norm	0.062	0.061	0.061	0.252	6.839	-3300.149	6616.419	
SH	0.049	0.048	0.048	0.201	6.869	-3247.827	6511.775	
t	0.049	0.048	0.048	0.204	6.924	-3242.680	6505.546	
GSH	0.049	0.048	0.048	0.197	6.782	-3246.696	6517.657	
SGSH	0.057	0.057	0.057	0.162	5.656	-3239.774	6503.812	

Table 7: Goodness-of-fit for different copulas IV

Copula	Parameters of the margins				Copula parameter			
	$\widehat{\mu}_1$	$\hat{\delta}_1$	$\widehat{\mu}_2$	$\hat{\delta}_2$	cp_1	cp_2	cp_3	$\mathcal{LL}$
$B0$	-0.014	1.056	-0.014	1.229	0.000	–	–	-3699.4
$B1$	-0.021	1.054	-0.025	1.222	0.708	–	–	-3290.9
$B2$	-0.015	1.084	-0.019	1.257	12.574	–	–	-3281.9
$B3$	-0.016	1.091	-0.020	1.271	6.748	–	–	-3274.8
$B4$	0.022	1.123	0.027	1.285	1.591	–	–	-3374.6
$B5$	-0.042	1.085	-0.052	1.278	1.897	–	–	-3485.9
$B4'$	-0.056	1.056	-0.067	1.242	0.933	–	–	-3469.8
$B6$	0.011	1.091	0.014	1.280	1.855	–	–	-3377.3
$B6'$	-0.028	1.101	-0.027	1.264	2.038	–	–	-3300.9
$B7$	0.015	1.099	0.018	1.291	1.134	–	–	-3394.0
$B8$	0.009	1.101	0.016	1.306	1.316	–	–	-3463.4
$B12$	0.076	1.029	0.089	1.184	0.992	–	–	-3381.9
$BB1$	-0.004	1.093	0.003	1.267	0.728	1.434	–	-3320.2
$BB3$	-0.008	1.072	0.001	1.246	1.506	0.280	–	-3357.8
$BB4$	0.021	1.119	0.026	1.280	1.577	0.088	–	-3374.3
$BB7$	-0.005	1.088	0.004	1.257	1.338	1.301	–	-3347.2
$BB10$	-0.024	1.078	-0.029	1.258	1.089	5.862	–	-3267.6
$BB11$	-0.012	1.063	-0.013	1.232	0.718	18.523	–	-3287.7
$BBMF$	-0.021	1.082	-0.027	1.261	1.056	6.488	–	-3269.6
$BC1$	-0.008	1.108	0.006	1.289	2.435	2.414	0.600	-3294.8
$BC2$	-0.008	1.113	0.006	1.293	3.227	3.286	0.407	-3300.1

Table 8: Normal margins: Estimated parameters

	Parameters of the margins						Copula parameter			
Copula	$\widehat{\mu}_1$	$\hat{\delta}_1$	$\hat{\nu}_1$	$\widehat{\mu}_2$	$\hat{\delta}_2$	$\hat{\nu}_2$	cp_1	cp_2	cp_3	$\mathcal{LL}$
$B0$	-0.036	0.89	6.91	-0.053	1.04	7.05	0.00	–	–	-3652.8
$B1$	-0.061	0.87	5.89	-0.077	1.01	5.96	0.72	–	–	-3225.4
$B2$	-0.047	0.91	7.28	-0.060	1.06	7.64	11.19	–	–	-3243.1
$B3$	-0.042	0.93	8.09	-0.054	1.09	8.12	6.22	–	–	-3241.0
$B4$	0.026	0.87	4.35	0.023	1.02	5.02	1.62	–	–	-3289.4
$B4'$	-0.141	0.80	3.70	-0.172	0.93	3.67	1.28	–	–	-3332.9
$B5$	-0.144	0.80	3.54	-0.175	0.93	3.52	2.14	–	–	-3348.6
$B6$	-0.094	0.84	4.37	-0.116	0.97	4.29	1.89	–	–	-3274.0
$B6'$	-0.031	0.88	5.22	-0.038	1.03	5.84	2.02	–	–	-3233.9
$B7$	-0.101	0.83	4.19	-0.124	0.96	4.11	1.17	–	–	-3275.4
$B8$	-0.113	0.81	3.82	-0.140	0.94	3.67	1.62	–	–	-3284.0
$B12$	0.045	0.84	5.65	0.047	0.98	6.36	1.00	–	–	-3321.3
$BB1$	-0.049	0.92	7.79	-0.061	1.07	7.73	1.20	5.51	–	-3229.1
$BB3$	-0.060	0.82	4.03	-0.072	0.96	4.21	1.66	0.31	–	-3238.9
$BB4$	0.026	0.87	4.35	0.023	1.02	5.02	1.62	0.00	–	-3289.4
$BB7$	-0.050	0.83	4.11	-0.062	0.97	4.32	1.66	1.27	–	-3241.1
$BB10$	-0.057	0.91	7.46	-0.070	1.06	7.43	1.21	4.48	–	-3231.6
$BB11$	-0.059	0.87	5.78	-0.073	1.01	5.90	0.72	19.53	–	-3223.6
$BBMF$	-0.049	0.92	7.79	-0.061	1.07	7.73	1.19	5.51	–	-3235.4
$BC1$	-0.039	0.87	5.18	-0.045	1.00	5.35	2.10	1.77	0.57	-3236.0
$BC2$	-0.040	0.88	5.15	-0.046	1.00	5.24	2.62	2.95	0.43	-3242.7

Table 9: $t(\nu)$ margins: Estimated parameters

Copula	Parameters of the margins $\widehat{\mu}_1$	$\hat{\delta}_1$	$\hat{t}_1$	$\widehat{\mu}_2$	$\hat{\delta}_2$	$\hat{t}_2$	Copula parameter cp_1	cp_2	cp_3	$\mathcal{LL}$
$B0$	-0.039	1.05	-0.73	-0.050	1.22	0.66	0.00	–	–	-3653.7
$B1$	-0.062	1.06	-1.12	-0.076	1.23	-0.82	0.71	–	–	-3227.9
$B2$	-0.047	1.07	-0.45	-0.059	1.23	0.98	11.27	–	–	-3243.7
$B3$	-0.041	1.07	0.78	-0.051	1.24	1.27	6.25	–	–	-3241.5
$B4$	0.020	1.13	-1.57	0.021	1.29	-1.32	1.62	–	–	-3291.9
$B4'$	-0.130	1.08	-1.57	-0.158	1.26	-1.57	1.19	–	–	-3344.7
$B5$	-0.130	1.09	-1.57	-0.159	1.28	-1.57	2.06	–	–	-3360.7
$B6$	-0.090	1.09	-1.57	-0.110	1.28	-1.57	1.88	–	–	-3280.4
$B6'$	-0.035	1.10	-1.39	-0.040	1.25	-0.92	2.02	–	–	-3235.7
$B7$	-0.097	1.09	-1.57	-0.118	1.27	-1.57	1.15	–	–	-3283.6
$B8$	-0.10	1.08	-1.57	-0.123	1.28	-1.57	1.55	–	–	-3300.3
$B12$	-0.042	1.02	-1.20	0.049	1.17	-0.52	0.99	–	–	-3322.7
$BB1$	-0.049	1.10	-1.56	-0.057	1.26	-1.36	0.70	1.50	–	-3233.5
$BB3$	-0.060	1.09	-1.57	-0.068	1.26	-1.47	1.64	0.29	–	-3247.4
$BB4$	0.018	1.13	-1.57	0.017	1.27	-1.19	1.58	0.00	–	-3292.2
$BB7$	-0.047	1.10	-1.57	-0.055	1.27	-1.57	1.61	1.28	–	-3247.6
$BB10$	-0.054	1.06	0.55	-0.065	1.23	1.06	1.18	4.70	–	-3232.7
$BB11$	-0.060	1.06	-1.15	-0.072	1.23	-0.86	0.72	20.10	0.00	-3226.1
$BBMF$	-0.060	1.09	-1.57	-0.068	1.26	-1.47	1.64	0.29	0.00	-3236.3
$BC1$	-0.040	1.10	-1.34	-0.044	1.25	-1.08	2.13	1.77	0.58	-3240.2
$BC2$	-0.041	1.10	-1.35	-0.044	1.26	-1.14	2.62	2.98	0.42	-3246.7

Table 10: GSH margins: Estimated parameters

Copula	Parameters of the margins								Copula parameter			
Copula	$\widehat{\mu}_1$	$\hat{\delta}_1$	$\hat{t}_1$	$\widehat{\mu}_2$	λ_1	$\hat{\delta}_2$	$\hat{t}_2$	λ_2	cp_1	cp_2	cp_3	$\mathcal{LL}$
$B0$	-0.19	1.05	-0.85	0.89	-0.30	1.20	0.83	0.86	0.00	–	–	-3643.0
$B1$	-0.19	1.05	-1.10	0.89	-0.29	1.20	-0.44	0.86	0.72	–	–	-3216.8
$B2$	-0.19	1.06	-0.54	0.90	-0.29	1.20	1.14	0.86	11.21	–	–	-3235.0
$B3$	-0.19	1.07	0.69	0.90	-0.29	1.22	1.39	0.87	6.23	–	–	-3232.6
$B4$	0.08	1.13	-1.57	1.05	0.06	1.30	-1.37	1.03	1.69	–	–	-3291.0
$B4'$	-0.39	1.08	-1.57	0.80	-0.50	1.24	-1.57	0.78	1.55	–	–	-3309.4
$B5$	-0.39	1.06	-1.57	0.80	-0.46	1.23	-1.57	0.80	2.34	–	–	-3323.3
$B6$	-0.34	1.07	-1.57	0.80	-0.48	1.23	-1.38	0.77	2.01	–	–	-3244.6
$B6'$	-0.02	1.10	-1.36	1.01	-0.07	1.25	-0.93	0.98	2.02	–	–	-3235.4
$B7$	-0.34	1.08	-1.57	0.80	-0.48	1.23	-1.42	0.77	1.30	–	–	-3245.8
$B8$	-0.35	1.07	-1.56	0.79	-0.49	1.23	-1.57	0.76	1.75	–	–	-3256.2
$B12$	-0.07	1.02	-1.25	0.91	-0.13	1.17	-0.48	0.89	0.99	–	–	-3315.5
$BB1$	-0.20	1.08	-1.54	0.89	-0.29	1.23	-1.26	0.86	0.44	1.64	–	-3225.9
$BB3$	-0.21	1.06	-1.57	0.88	-0.32	1.22	-1.48	0.84	1.70	0.16	–	-3236.5
$BB4$	0.09	1.13	-1.57	1.05	0.06	1.30	-1.34	1.02	1.70	0.03	–	-3291.0
$BB7$	-0.18	1.08	-1.57	0.90	-0.27	1.25	-1.56	0.87	1.80	1.03	–	-3241.5
$BB10$	-0.22	1.05	-0.53	0.88	-0.33	1.21	0.96	0.84	1.29	3.97	–	-3221.5
$BB11$	-0.19	1.06	-1.15	0.89	-0.29	1.20	-0.52	0.86	0.72	19.59	–	-3214.9
$BBMF$	-0.18	1.06	0.58	0.91	-0.28	1.21	1.29	0.87	1.19	5.54	–	-3228.6
$BC1$	-0.17	1.09	-1.36	0.91	-0.28	1.22	-0.99	0.86	2.04	1.89	0.49	-3233.6
$BC2$	-0.17	1.09	-1.37	0.91	-0.29	1.23	-1.05	0.86	2.73	2.91	0.51	-3239.8

Table 11: *SGSH* margins: Estimated parameters

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