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Abstract

A generalization of the hyperbolic secant distribution which allows both for skewness and for leptokurtosis was given by Morris (1982). Recently, Vaughan (2002) proposed another flexible generalization of the hyperbolic secant distribution which has a lot of nice properties but is not able to allow for skewness. For this reason, Fischer and Vaughan (2002) additionally introduced a skewness parameter by means of splitting the scale parameter and showed that most of the nice properties are preserved. We briefly review both classes of distributions and apply them to financial return data. By means of the Nikkei225 data, it will be shown that this class of distributions – the so-called skew generalized secant hyperbolic distribution – provides an excellent fit in the context of unconditional and conditional return models.

JEL classification: C22; G12

Keywords: SGSH distribution; NEF-GHS distribution; skewness; GARCH; APARCH

1 Introduction

The hyperbolic secant distribution — which was first studied by Baten (1934) and Talacko (1956) — seems to be an appropriate candidate as a starting point for financial return models: Firstly, it exhibits more leptokurtosis than the normal and even more than the logistic distribution. Secondly, the cumulative distribution function admits a closed form implying that, for example, risk neutral probabilities of option prices can be calculated fast and accurate. Thirdly, this distribution is reproductive (i.e. the class is preserved under convolution), infinitely divisible with existing moment-generating function and has finite moments. Since 1956, two generalization have been proposed which both incorporate most of these properties, too and, in addition, allow for a more flexible form concerning skewness and leptokurtosis.

The first generalization was proposed by Morris (1982) in the context of natural exponential families (NEF) with quadratic variance function (i.e. the variance is a quadratic function of the mean). In this class, consisting of six members, one distribution — the so-called *NEF-GHS distribution* — is generated by the hyperbolic secant distribution. The NEF-GHS distribution allows for skewness and arbitrarily high excess kurtosis. Morris (1982) showed that this class is again reproductive, infinitely divisible with existing moment-generating function and existing moments. However, the corresponding cumulative distribution function doesn't admit a closed form.

Recently, Vaughan (2002) suggested a family of symmetric distributions — the so-called *generalized secant hyperbolic (GSH) distribution* — with kurtosis ranging from 1.8 to infinity. This family includes both the hyperbolic secant and the logistic distribution and closely approximates the Student t-distribution with corresponding kurtosis. In addition, the moment-generating function and all moments exist, and the cumulative distribution is given in closed form. Unfortunately, this family does not allow for skewness. For this purpose, Fischer and Vaughan (2002) introduced a skewness parameter by means of splitting the scale parameter according to Fernandez, Osiewalski and Steel (1995). This method preserves the closed form for the density, the cumulative distribution function, the inverse cumulative distribution function.

It will be shown that this family — termed as *skew generalized secant hyperbolic distribution* (SGSH) — provides an excellent fit to the Nikkei225 data. This is verified in the context of unconditional and conditional return models. In particular, we compare the results to other popular return models which have been proposed in the literature in the past: The α-stable distributions (see, for example, Mittnik et al. (1998)), the class of generalized hyperbolic distributions (see, for example, Prause (1999)), the generalized logistic family of McDonald (1991) and the skewed generalized t-distribution of the second kind of Grottke (2002).

2 Generalizations of Hyperbolic Secant Distributions

2.1 (Generalized) Hyperbolic Secant distribution

A symmetric random variable X is said to follow a *hyperbolic secant* (HS) *distribution* if its probability density function (with unit variance) is given by

$$f_{HS}(x) = \frac{1}{2\cosh(\pi x/2)}$$

or, equivalently, its cumulative distribution function is given by

$$F_{HS}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(\sinh(\pi x/2)).$$

This distribution is more leptokurtic than the normal, even more leptokurtic than the logistic distribution and has a kurtosis coefficient (measured by the fourth standardized moment) of 5. Consequently, it seems to be a reasonable "starting point" as a distribution for leptokurtic data, in particular for financial return data. In order to obtain higher "leptokurtic flexibility", the λ-th convolution of a hyperbolic secant distribution can be considered. This was discussed, for example, by Harkness and Harkness (1968) or Jørgensen (1997). The resulting distribution is commonly known as *generalized hyperbolic secant* (GHS) *distribution*. However, GHS offers still no opportunity to take skewness into account.

2.2 NEF-GHS Distribution

The NEF-GHS distribution was originally introduced by Morris (1982) in the context of *natural exponential families* (NEF) with specific quadratic variance functions. Densities of natural exponential families are of the form

$$f(x;\lambda,\theta) = \exp\{\lambda x - \psi(\lambda,\theta)\}\cdot\zeta(x,\lambda). \tag{1}$$

In the case of the NEF-GHS distribution, $\psi(\lambda,\theta) = -\lambda\log(\cos(\theta))$ and $\zeta(x,\lambda)$ equals the probability density function of a generalized hyperbolic secant (GHS) distribution. Hence, the probability density function of the NEF-GHS distribution is given by

$$f(x;\lambda,\theta) = \underbrace{\frac{2^{\lambda-2}}{\pi\,\Gamma(\lambda)}\cdot\left|\Gamma\left(\frac{\lambda+ix}{2}\right)\right|^2}_{C(x)}\cdot\exp\{\theta x + \lambda\log(\cos(\theta))\} \tag{2}$$

for $\lambda > 0$ and $|\theta| < \pi/2$. Introducing a scale parameter $\delta > 0$ and a location parameter $\mu \in \mathbb{R}$, and setting $\beta \equiv \tan(\theta) \in \mathbb{R}$, equation (2) changes to

$$f(x) = C\left(\frac{x-\mu}{\delta}\right) \cdot \exp\left(\arctan(\beta) \cdot \frac{x-\mu}{\delta} + \lambda \log(\cos(\arctan(\beta)))\right). \tag{3}$$

The NEF-GHS distribution is a flexible class of distribution which allows for skewness and excess kurtosis, which is infinitely divisible with existing moment-generating function and hence, existing moments. However, the cumulative distribution function and the inverse cumulative distribution function is not available in a closed form. These properties facilitate calculating risk measures, constructing multivariate copula-based distributions (see, for example, Fischer (2003)) or valuating option pricing models (see Fischer (2002)).

2.3 GSH distribution

Another generalization of the hyperbolic secant distribution – which is able to model both thin and fat tails – was introduced by Vaughan (2002). This distribution family, the so-called standard *generalized secant hyperbolic* (GSH) distribution with kurtosis parameter $t \in (-\pi, \infty)$, has density

$$f_{GSH}(x;t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t)\exp(c_2(t)x) + 1}, \qquad x \in \mathbb{R} \tag{4}$$

with

$$\begin{array}{llll} a(t) = \cos(t), & c_2(t) = \sqrt{\frac{\pi^2 - t^2}{3}} & c_1(t) = \frac{\sin(t)}{t} \cdot c_2(t), & \text{for } -\pi < t \leq 0, \\ a(t) = \cosh(t), & c_2(t) = \sqrt{\frac{\pi^2 + t^2}{3}} & c_1(t) = \frac{\sinh(t)}{t} \cdot c_2(t), & \text{for } t > 0 \end{array}.$$

The density from (4) is chosen so that X has zero mean and unit variance. The GSH distribution includes the logistic distribution ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as limiting case for $t \to \infty$. Vaughan (2002) derives the cumulative distribution function, depending on the parameter t, as

$$F_{GSH}(x;t) = \begin{cases} 1 + \frac{1}{t}\operatorname{arccot}\left(-\frac{\exp(c_2(t)x)+\cos(t)}{\sin(t)}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\exp(\pi x/\sqrt{3})}{1+\exp(\pi x/\sqrt{3})} & \text{for } t = 0, \\ 1 - \frac{1}{t}\operatorname{arccoth}\left(\frac{\exp(c_2(t)x)+\cosh(t)}{\sinh(t)}\right) & \text{for } t > 0. \end{cases}$$

and the inverse cumulative distribution function

$$F^{-1}_{GSH}(u;t) = \begin{cases} \frac{1}{c_2(t)} \ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi} \ln\left(\frac{u}{1-u}\right) & \text{for } t = 0, \\ \frac{1}{c_2(t)} \ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \text{for } t > 0. \end{cases}$$

The moment-generating function also depends on t and is given by

$$\mathcal{M}_{GSH}(u;0,1,t) = \begin{cases} \frac{\pi}{t}\sin(ut/c_2(t))\csc(u\pi/c_2(t)) & \text{für } t \in (-\pi, 0), \\ \sqrt{3}u\csc(\sqrt{3}u) & \text{für } t = 0, \\ \frac{\pi}{t}\sinh(ut/c_2(t))\csc(u\pi/c_2(t)) & \text{für } t > 0. \end{cases}$$

Moments of X can be deduced from the last equation. Despite of its nice properties, the GSH distributions is not able to include skewness effects.

2.4 SGSH distributions

There are plenty of methods in the literature to make a symmetric distribution skew. As the cumulative distribution function of the GSH distribution is explicitly known, Fischer and Vaughan (2002) decided in favour of splitting the scale parameter, as it was done by Fernandez, Osiewalski and Steel (1995) for the skewed exponential power distribution.

Let $\gamma > 1$, $\mathbf{I}^+(x)$ denote the indicator function for x on $\mathbb{R}_+$ and $\mathbf{I}^-(x)$ denote the indicator function for x on $\mathbb{R}_-$. Then it can be easily verified that

$$f_{SGSH}(x;t,\gamma) = \frac{2}{\gamma + \frac{1}{\gamma}} \left\{ f_{GSH}(x/\gamma) \cdot \mathbf{I}^-(x) + f_{GSH}(\gamma x) \cdot \mathbf{I}^+(x) \right\}$$

$$= \frac{2c_1}{\gamma + \frac{1}{\gamma}} \cdot \left(\frac{\exp(c_2 x/\gamma) \cdot \mathbf{I}^-(x)}{\exp(2c_2 x/\gamma) + 2a\exp(c_2 x/\gamma) + 1} + \frac{\exp(c_2 \gamma x) \cdot \mathbf{I}^+(x)}{\exp(2c_2 \gamma x) + 2a\exp(c_2 \gamma x) + 1} \right)$$

is a density function which is symmetric for $\gamma = 1$, skewed to the right for $\gamma > 1$ and skewed to the left for $0 < \gamma < 1$. The corresponding distribution is termed as *skew generalized secant hyperbolic* (SGSH) *distribution* in the sequel. The effect of the skewness parameter on the density can be seen in figure 1, below.

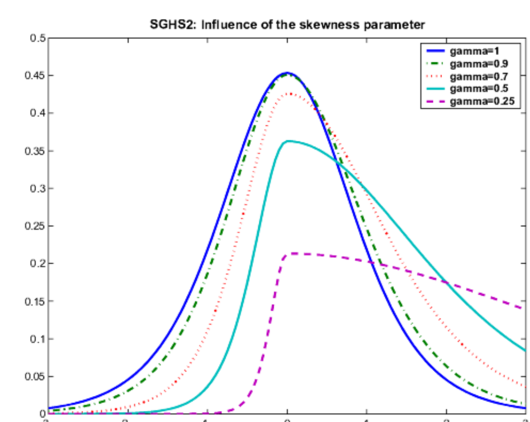


Figure 1: Effect of the skewness parameter.

It can also be shown (see, for example, Grottke (2002), p. 21) that the cumulative and the inverse cumulative distribution functions admits a closed form, namely

$$F_{SGSH}(x;t,\gamma) = \frac{2\gamma^2}{\gamma^2+1} \cdot \left(F_{GSH}\left(x/\gamma\right) \cdot \mathbf{I}^{-}(x) + \left(\frac{\gamma^2 - 1 + 2F_{GSH}(\gamma x)}{2\gamma^2} \right) \cdot \mathbf{I}^{+}(x) \right),$$

$$F^{-1}_{SGSH}(x;t,\gamma) = \gamma F^{-1}_{GSH}\left(x \cdot \frac{\gamma^2+1}{2\gamma^2} \right) \mathbf{I}^{A}(x) + \frac{1}{\gamma} F^{-1}_{GSH}\left(x \cdot \frac{\gamma+1}{2} - \frac{\gamma-1}{2} \right) \mathbf{I}^{\overline{A}}(x),$$

with

$$\mathbf{I}^{A}(x) = \begin{cases} 1, & \text{for } x < \frac{\gamma^2}{1+\gamma}, \\ 0, & \text{for } x \geq \frac{\gamma^2}{1+\gamma}. \end{cases} \qquad \text{and} \qquad \mathbf{I}^{\overline{A}}(x) = 1 - \mathbf{I}^{A}(x).$$

In addition, the power moments of a SGHS distribution can be deduced from that of a GHS distribution by means of

$$\mathbf{E}(X^r_{SGSH_2}) = \mathbf{E}^{+}(X^r_{GSH}) \cdot \frac{2\gamma}{\gamma^2+1} \cdot \left[(-1)^r \gamma^{r+1} + \gamma^{-r-1} \right], \tag{5}$$

with the partial positive expectation value

$$\mathbf{E}^{+}(X^r_{GSH}) = \int_0^\infty x^r f_{GSH}(x) dx.$$

Note, that for *even* r the positive half moments $\mathbf{E}^{+}(X^r_{GSH})$ can be deduced from $\mathbf{E}(X^r_{GSH})$ by division with 2: Setting $\gamma = 1$ in (5),

$$\mathbf{E}(X^r_{GSH}) = ((-1)^r + 1) \cdot \mathbf{E}^{+}(X^r_{GSH}).$$

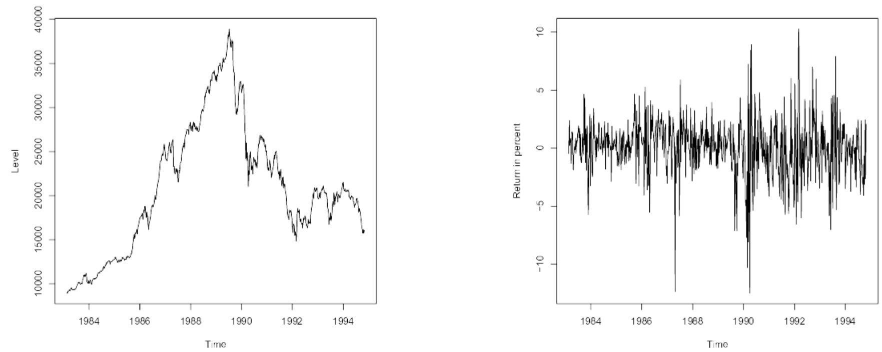
For *odd* r, the formula for the half moments $\mathbf{E}^{+}(X^r_{GSH})$ is slightly more complicated. The corresponding results are deduced by Fischer and Vaughan (2002).

3 Financial Application of the SGSH distribution

3.1 The data set

In order to adopt and compare estimation results for a great deal of distributions – in particular the stable distributions (STABLE) – priority is given to the weekly returns of the Nikkei from July 31, 1983 to April 9, 1995, with $N = 608$ observations. This series was intensively investigated, for example, by Mittnik, Paolella and Rachev (1998) because it exhibits typical stylized facts of financial return data. Figure 2 illustrates the time series of levels and corresponding log-returns.

Figure 2: Levels and returns of Nikkei.



(a) Levels (b) Returns

3.2 Unconditional fit to financial return data

Similar to Mittnik, Paolella and Rachev (1998), four criteria are employed to compare the goodness-of-fit of the different candidate distributions. The first is the *log-Likelihood value* ($\mathcal{LL}$) obtained from the Maximum-Likelihood estimation. The $\mathcal{LL}$-value can be considered as an "overall measure of goodness-of-fit and allows us to judge which candidate is more likely to have generated the data". As distributions with different numbers of parameters k are used, this is taken into account by calculating the *Akaike criterion* given by

$$AIC = -2 \cdot \mathcal{LL} + \frac{2N(k+1)}{N-k-2}.$$

The third criterion is the *Kolmogorov-Smirnov distance* as a measure of the distance between the estimated parametric cumulative distribution function, $\hat{F}$, and the empirical sample distribution, F_{emp}. It is usually defined by

$$\mathcal{K} = 100 \cdot \sup_{x \in \mathbb{R}} |F_{emp}(x) - \hat{F}(x)|. \tag{6}$$

Finally, the *Anderson-Darling statistic* is calculated, which weights $|F_{emp}(x) - \hat{F}(x)|$ by the reciprocal of the standard deviation of F_{emp}, namely $\sqrt{\hat{F}(x)(1-\hat{F}(x))}$, that is

$$\mathcal{AD}_0 = \sup_{x \in \mathbb{R}} \frac{|F_{emp}(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}}. \tag{7}$$

Instead of just the maximum discrepancy, the second and third largest value, which is commonly termed as $\mathcal{AD}_1$ and $\mathcal{AD}_2$, are also taken into consideration. Whereas $\mathcal{K}$ emphasizes deviations around the median of the fitted distribution, $\mathcal{AD}_0, \mathcal{AD}_1$ and $\mathcal{AD}_2$ allow discrepancies in the tails of the distribution to be appropriately weighted.

Distribution	k	$\mathcal{LL}$	**AIC**	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
NORM	2	-1428.3	2862.6	6.89	4.920	2.810	1.070
STABLE	4	-1393.2	2796.5	3.00	0.085	0.084	0.081
HS	2	-1393.4	2792.8	4.31	0.216	0.150	0.121
GHS	3	-1392.2	2794.6	4.15	0.140	0.117	0.114
NEF-GHS	4	-1388.1	2786.3	2.42	0.091	0.090	0.083
GSH	3	-1392.3	2794.8	4.17	0.142	0.117	0.114
SGSH	4	-1387.5	**2785.2**	2.18	0.088	0.087	0.080
LOG	2	-1398.1	2802.1	4.56	0.362	0.236	0.186
EGB2	4	-1388.1	2786.3	2.45	0.103	0.100	0.095
GH	5	-1388.0	2788.2	2.43	0.095	0.093	0.086
HYP	4	-1388.2	2786.5	2.50	0.106	0.103	0.098
NIG	4	-1388.2	2786.6	2.48	0.085	0.085	0.075
SGT2	5	**-1387.4**	2786.9	**2.12**	0.076	0.072	0.071
Student-t	3	-1392.2	2792.5	3.77	0.107	0.104	0.103
gh-NORM	4	-1388.7	2788.5	2.27	**0.068**	**0.062**	**0.061**

Table 1: Goodness-of-fit for the unconditional case: Nikkei225.

Estimation was performed not only for the two families of generalized hyperbolic secant distributions (NEF-GHS and SGSH with symmetric counterparts GHS and GSH), but also for distribution families which have become popular in finance in the last years: Firstly, the generalized hyperbolic (GH) distributions which were discussed by Prause (1999) and include, for example, the Normal-inverse Gaussian (NIG) distributions (see Barndorff-Nielsen (1997)) as well as the hyperbolic (HYP) distributions (see Eberlein and Keller (1995)) as special cases. Secondly, the exponential generalized beta of the second kind (EGB2) distribution that was introduced by McDonald (1991) as a generalization of the logistic (LOG) distribution and used in various financial applications, see also Fischer (2002). Thirdly, a very flexible skew version of the generalized t-distribution (SGT2) proposed by Grottke (2001). Finally, we performed calculations for the gh-transformed normal (gh-NORM) distribution (see Klein and Fischer (2002)).

The estimation results are summarized in table 1, above and are as follows: Firstly, let us focus on the fit of generalized hyperbolic secant families. There seems to be no difference between the GSH distribution of Vaughan (2002) and the GHS distribution of Harkness and Harkness (1968). This is not true if we consider the skewed pendants and compare the NEF-GHS distribution of Morris (1982) with the SGSH distribution of Fischer and Vaughan (2002) which exhibits better goodness-of-fit values with respect to all five criteria. For that reason, we restrict our considerations to the SGHS distribution. Concerning the $\mathcal{LL}$-value, only the SGT2 distribution has a (slightly) higher value. The same is true if we compare the $\mathcal{K}$-values. If we take the number of parameters into account (i.e. focus on the AIC criterion), SGSH even outperforms SGT2. The situation is a little bit different concerning the tail fit. Here, gh-transformed distributions finished best, followed by SGT2, NIG, STABLE and SGHS (Note, that the last three are close together).

3.3 Conditional fit to financial return data

Assuming independent observations – as we did it in the last subsection – is not very realistic. To capture dependency between different log-returns, generalized autoregressive conditionally heteroscedastic (GARCH) models have proposed by Engle (1982) and Bollerslev (1986) as models for financial return data. These models are able to capture the distributional stylized facts (like thick tails or high peakedness), on the one hand, as well as the time series stylized facts (like volatility clustering), on the other hand. The setting for our GARCH framework is similar to Bollerslev (1986) assuming that the log-returns R_t of financial data are given by

$$\Theta_m(L)R_t = \mu + U_t$$

with

$$U_t|F_{t-1} \sim D(0, h_t^2, \eta) \text{ or } U_t|F_{t-1} = h_t\epsilon_t \text{ with } \epsilon_t \sim D(0, 1, \eta),$$

where $\Theta_m(L)$ is a polynomial in the lag operator L of order m. For reasons of simplicity, assume that $\Theta_m(L) \equiv 1$ and $\mu \equiv 0$. The residuals $\{U_t\}$ are assumed to follow a GARCH-D process. That means they follow a distribution[1] D with shape parameter η and time-varying variance h_t^2. In the GARCH$(1,1)$-Normal specification from Bollerslev (1986) it is given by

$$h_t^2 = \alpha_0 + \alpha_1 R_{t-1}^2 + \beta_1 h_{t-1}^2 = \alpha_0 + \alpha_1 h_{t-1}^2 \epsilon_{t-1}^2 + \beta_1 h_{t-1}^2. \tag{8}$$

Note, that setting $\beta_1 = 0$ results in the ARCH model of Engle (1982). The estimation results for the standard GARCH setting are summarized in table 2, below.

[1] Although GARCH models with conditionally normally distributed errors imply unconditionally leptocurtic distributions, there is evidence (see, for example, Bollerslev, 1987) that starting with leptocurtic and possibly skewed (conditional) distribution will achieve better results. For that reason, alternative error distributions are used.

D	k	$\mathcal{LL}$	AIC	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
NORM	4	-1358.7	2727.8	4.31	505.30	0.890	0.141
STABLE	6	-1340.8	2695.8	3.76	0.253	0.092	0.093
HS	4	-1342.8	2695.6	4.54	0.427	0.126	0.125
GHS	5	-1342.3	2698.7	4.01	0.679	0.122	0.122
NEF-GHS	6	-1334.6	2683.4	2.25	0.461	0.083	0.076
GSH	5	-1341.3	2696.8	3.71	0.914	0.155	0.116
SGSH	6	1333.0	2680.2	**1.98**	0.504	0.069	0.068
LOG	4	-1344.0	2698.2	3.55	0.966	0.151	0.112
EGB2	6	-1335.6	2685.4	2.60	0.973	0.086	0.062
GH	7	-1331.3	2678.8	2.27	**0.150**	0.062	0.061
HYP	6	-1333.3	2680.9	2.21	0.834	0.079	0.063
NIG	6	-1332.5	2679.1	2.19	0.466	0.065	0.064
SGT2	7	**-1331.1**	**2678.4**	2.32	0.172	**0.057**	**0.056**
Student-t	5	-1340.3	2692.7	3.81	0.267	0.119	0.118

Table 2: Goodness-of-fit for GARCH(1,1)-models: Nikkei225.

Again, the SGSH distribution outperforms most of its competitors and even has the lowest $\mathcal{K}$-value. Concerning the $\mathcal{LL}$-value, only SGT2 and GH produced slightly higher values. They same is true for $\mathcal{AD}_0$.

GARCH models have been generalized in many different ways. In order to capture leverage effects, Zakoian (1994) proposed the threshold (T)-GARCH model with standard deviation given by

$$h_t = \alpha_0 + \alpha_1^+ R_{t-1}^+ - \alpha_1^- R_{t-1}^- + \beta_1 h_{t-1}, \tag{9}$$

where $R_t^+ = \max\{R_t, 0\}$ and $R_t^- = \min\{R_t, 0\}$. Imposing a Box-Cox-transformation on the conditional standard deviation process and the asymmetric absolute returns leads to the asymmetric power (AP-ARCH) specification of Ding, Engle and Granger (1993), where the variance equation is

$$h_t^\lambda = \alpha_0 + \alpha_1(|R_{t-1}| - cR_{t-1})^\lambda + \beta_1 h_{t-1}^\lambda. \tag{10}$$

Equation (10) reduces to (9) for $\lambda = 1$, $\alpha_1 = \alpha_1^-/(2-\alpha_1^+)$ and $c = 1-\alpha_1^+(2-\alpha_1^+)/\alpha_1^-$. Moreover, equation (8) is achieved for $\lambda = 2$ and $c = 0$.

To take also asymmetric effects into account, we end up with an AP-ARCH(1,1)-D specification whose estimation results are summarized in table 3. Note that AP-

ARCH(1,1) estimation results for stable distribution were not available. Moreover, $**$ indicates convergence problems of our algorithm.

D	**k**	$\mathcal{LL}$	**AIC**	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
NORM	6	-1352.8	2719.8	4.28	1021.57	0.473	0.125
HS	6	-1339.7	2693.5	3.98	0.489	0.133	0.128
GHS	7	-1338.6	2695.5	3.52	1.045	0.119	0.120
NEF-GHS	8	-1333.7	2685.7	2.08	0.500	0.079	0.077
GSH	7	-1337.3	2693.0	3.31	1.093	0.117	0.115
SGSH	8	1331.8	2681.9	**1.89**	0.543	0.073	0.067
LOG	6	-1339.6	2693.4	3.08	1.183	0.113	0.112
EGB2	8	-1334.6	2687.5	2.46	1.011	0.078	0.077
GH	9	**	**	**	**	**	**
HYP	8	-1332.5	2683.2	2.09	0.899	0.078	0.073
NIG	8	-1331.6	2681.8	2.07	0.502	0.087	0.075
SGT2	9	**-1329.8**	**2679.9**	2.26	**0.192**	**0.065**	**0.062**
Student-t	7	-1336.1	2688.3	3.28	0.398	0.108	0.105

Table 3: Goodness-of-fit for AP-ARCH(1,1)-models: Nikkei225.

It can be observed that for most of the distributions the $\mathcal{K}$-values improve (compared to the GARCH(1,1)-fit) whereas the $\mathcal{AD}$-values become worser. SGT2 and SGSH seem to dominate the other distributions. Except of $\mathcal{AD}_0$, SGSH is close to SGT2. Again, SGSH achieves the smallest $\mathcal{K}$-value.

4 Conclusions

Two generalizations of the hyperbolic secant distribution have been proposed in the last years which seem to be encouraging as a model for financial return data: The NEF-GHS distribution of Morris (1982) and the SGSH distribution of Fischer and Vaughan (2002). Both incorporate skewness and leptokurtosis. Within this work we applied them to the weekly returns of the Japanese stock index Nikkei225. Firstly, results of the unconditional fit (or of the GARCH(0,0)-D model) were calculated (for a broad class of distributions D which were established as suitable return models in the literature, and for the generalized hyperbolic secant families) as a benchmark. In a second step, volatility cluster were taken into account by means of a GARCH(1,1)-D model. Finally, we also tried to model leverage effects and estimated an AP-

ARCH(1,1)-D model. For the Nikkei data, we found that the skew generalized secant hyperbolic (SGHS) distribution provides an excellent fit in all cases. It dominates the NEF-GHS distribution as well as the EGB2 distribution. Furthermore, it approves as flexible but numerically easier to implement as the generalized hyperbolic family. Only SGT2 – for which neither the cumulative distribution function nor the inverse cumulative distribution function are known – slightly outperforms SGSH.

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