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Classes of Skew Generalized Hyperbolic Secant Distributions

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Abstract

A generalization of the hyperbolic secant distribution which allows both for skewness and for leptokurtosis was given by Morris (1982). Recently, Vaughan (2002) proposed another flexible generalization of the hyperbolic secant distribution which has a lot of nice properties but is not able to allow for skewness. For that reason, we additionally introduce a skewness parameter by means of splitting the scale parameter and show that most of the nice properties are preserved. Finally, we compare both families with respect to their ability to model financial return distributions.

Keywords: Skewed hyperbolic secant; NEF-GHS distribution; GSH distribution; skewness; return data

1 Preface

The hyperbolic secant distribution, first by Baten (1934) and by Talacko (1956), seems to be an appropriate candidate as a starting point for a model for financial return data: Firstly, it exhibits more leptokurtosis than the normal and even more than the logistic distribution. Secondly, the cumulative distribution function admits a closed form so that, for example, risk neutral probabilities of option prices can be calculated quickly and accurately. Thirdly, this distribution is self-conjugate, infinitely divisible with existing moment-generating function and finite moments. Since 1956, two generalization have been proposed, both of which incorperate most of these properties, and, in addition, allow for a more flexible form in relation to skewness and leptokurtosis.

The first proposal was introduced by Morris (1982) in the context of natural exponential families (NEF) with quadratic variance function (i.e. the variance is a quadratic function of the mean). This class has six members, one of which is generated by the hyperbolic secant distribution, the so-called NEF-GHS distribution. The NEF-GHS distribution allows for skewness and arbitrarily high excess kurtosis. Morris showed that this class is again reproductive, infinitely divisible with existing moment-generating function and existing moments. However, the corresponding cumulative distribution function doesn't admit a closed form.

Recently, Vaughan (2002) proposed a family of symmetric distributions, which he called the *generalized secant hyperbolic (GSH) distribution*, with kurtosis ranging from 1.8 to infinity. This family includes both the hyperbolic secant and the logistic distribution and closely approximates the Student t-distribution with corresponding kurtosis. In addition, the moment-generating function and all moments exist, and the cumulative distribution is given in closed form. Unfortunately, this family does not allow for skewness. For this purpose, we introduce a skewness parameter by means of splitting the scale parameter according to Fernandez, Osiewalski and Steel (1995). This transformation preserves the closed form for the density, the cumulative distribution function, the inverse cumulative distribution function and the moments. We show that this family, termed as *skewed generalized secant hyperbolic distribution*, provides an excellent fit to the Nikkei225 data set.

2 The hyperbolic secant distribution

A symmetric random variable X is said to follow a *hyperbolic secant distribution* (HS) if its probability density function (with unit variance and 0 mean) is given by

$$f_{HS}(x) = \frac{1}{2\cosh(\pi x/2)}$$

or, equivalently, its cumulative distribution function is given by

$$F_{HS}(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(\sinh(\pi x/2)).$$

This distribution is more leptokurtic than the normal, even more leptokurtic than the logistic distribution and has a kurtosis coefficient (measured by the fourth standardized moment) of 5. Moreover, the hyperbolic secant is infinitely divisible (see, for example, Feller (1971)) with existing moment-generating function $\mathcal{M}_{HS}(u) = 1/\cos(u)$. Consequently, it seems to be a suitable starting point as a distribution for leptokurtic data, in particular financial return data. One possibility to allow for more "leptokurtic flexibility", is to consider the λ-th convolution of a hyperbolic secant distribution. The latter was discussed, for example, by Harkness and Harkness (1968) or Jørgensen (1997), and is commonly known as the *generalized hyperbolic secant distribution* (GHS). However, GHS offers still no opportunity to take skewness into account.

3 The NEF-GHS distribution

The NEF-GHS distribution was originally introduced by Morris (1982) in the context of *natural exponential families* (NEF) with specific quadratic variance functions. Densities of natural exponential families are of the form

$$f(x;\lambda,\theta) = \exp\{\theta x - \psi(\lambda,\theta)\} \cdot \zeta(x,\lambda). \tag{3.1}$$

In the case of the NEF-GHS distribution, $\psi(\lambda,\theta) = -\lambda\log(\cos(\theta))$ and $\zeta(x,\lambda)$ equals the probability density function of a generalized hyperbolic secant (GHS) distribution. Hence, the probability density function of the NEF-GHS distribution is given by

$$f(x;\lambda,\theta) = \underbrace{\frac{2^{\lambda-2}}{\pi\,\Gamma(\lambda)} \cdot \left|\Gamma\left(\frac{\lambda+ix}{2}\right)\right|^2}_{C(x)} \cdot \exp\left\{\theta x + \lambda\log(\cos(\theta))\right\} \tag{3.2}$$

for $\lambda > 0$ and $|\theta| < \pi/2$. Introducing a scale parameter $\delta > 0$ and a location parameter $\mu \in \mathbb{R}$, and setting $\beta \equiv \tan(\theta) \in \mathbb{R}$, equation (3.2) changes to

$$f(x) = C\left(\frac{x-\mu}{\delta}\right) \cdot \exp\left(\arctan(\beta) \cdot \frac{x-\mu}{\delta} + \lambda\log(\cos(\arctan(\beta)))\right). \tag{3.3}$$

It can be shown that the NEF-GHS distribution reduces to the GHS distribution for $\theta = \beta = 0$; to a skewed hyperbolic secant distribution for $\lambda = 1$; and to the hyperbolic secant distribution for $\lambda = 1$ and $\theta = 0$. Furthermore, it goes in limit to the normal distribution ($\lambda \to \infty$). It was shown that a NEF-GHS variable X is infinitely divisible and reproductive. Furthermore the moment-generating function of X exists for $\{u| \cos(u) - \beta \sin(u) > 0\}$ and is given by

$$\mathcal{M}(u) = \exp\{-\lambda \log(\cos(u) - \beta \sin(u))\}. \tag{3.4}$$

All moments exist. In particular, the range of $\mathbb{S}(X)$ and $\mathbb{K}(X)$ is unrestricted. From (3.4), the first four moments $m'_i = \mathbf{E}(X^i)$, $i = 1, .., 4$ are given by

$$m'_1 = \mathbf{E}(X) = \mu + \delta\lambda\beta,$$

$$m'_2 = \delta^2\, \lambda(\beta^2 + 1 + \lambda\beta^2),$$

$$m'_3 = \delta^3\, \lambda(\beta^3\lambda^2 + 3\lambda\beta + 2\beta + 3\lambda\beta^3 + 2\beta^3) \text{ and}$$

$$m'_4 = \delta^4\, \lambda(2 + 3\lambda + 6\beta^4 + 8\beta^2 + 11\lambda\beta^4 + 6\lambda^2\beta^2 + 14\lambda\beta^2 + \lambda^3\beta^4 + 6\lambda^2\beta^4).$$

Consequently, the corresponding central moments $m_i = \mathbf{E}(X - m)^i$ are

$$\mathbf{V}(X) = m_2 = \lambda(1 + \beta^2),$$

$$m_3 = 2\delta^3\lambda\beta(\beta^2 + 1) \text{ and } m_4 = \delta^4[3\lambda(\lambda + 2)(1 + \beta^2)^2 - 4\lambda(1 + \beta^2)].$$

The skewness and kurtosis coefficients, measured by the third and fourth standardized moments, respectively, are given by

$$\mathbb{S}(X) = \frac{m_3}{\sqrt{(m_2)^3}} = \frac{2\beta}{\sqrt{\lambda(1 + \beta^2)}} \quad \text{and} \quad \mathbb{K}(X) - 3 = \frac{m_4}{(m_2)^2} - 3 = \frac{2 + 6\beta^2}{\lambda(1 + \beta^2)}.$$

4 The GSH distributions

Another generalization of the hyperbolic secant distribution, one that can model both thin and fat tails, was introduced by Vaughan (2002). This so-called standard *generalized* *secant hyperbolic* (GSH) family of distributions, with kurtosis parameter $t \in (-\pi, \infty)$, has density

$$f_{GSH}(x;t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t)\exp(c_2(t)x) + 1}, \qquad x \in \mathbb{R} \tag{4.5}$$

with

$$a(t) = \cos(t), \quad c_2(t) = \sqrt{\tfrac{\pi^2 - t^2}{3}} \quad c_1(t) = \tfrac{\sin(t)}{t} \cdot c_2(t), \quad \text{for } -\pi < t \leq 0,$$
$$a(t) = \cosh(t), \quad c_2(t) = \sqrt{\tfrac{\pi^2 + t^2}{3}} \quad c_1(t) = \tfrac{\sinh(t)}{t} \cdot c_2(t), \quad \text{for } t > 0.$$

The density in (4.5) is chosen so that X has zero mean and unit variance. The GSH distribution includes the logistic distribution ($t = 0$) and the hyperbolic secant distribution ($t = -\pi/2$) as special cases and the uniform distribution on $(-\sqrt{3}, \sqrt{3})$ as the limiting case for $t \to \infty$. Vaughan derives the cumulative distribution function, depending on the parameter t, as

$$F_{GSH}(x;t) = \begin{cases} 1 + \frac{1}{t}\operatorname{arccot}\left(-\frac{\exp(c_2(t)x)+\cos(t)}{\sin(t)}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\exp(\pi x/\sqrt{3})}{1+\exp(\pi x/\sqrt{3})} & \text{for } t = 0, \\ 1 - \frac{1}{t}\operatorname{arccoth}\left(\frac{\exp(c_2(t)x)+\cosh(t)}{\sinh(t)}\right) & \text{for } t > 0. \end{cases}$$

and the inverse distribution function

$$F^{-1}_{GSH}(u;t) = \begin{cases} \frac{1}{c_2(t)}\ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \text{for } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi}\ln\left(\frac{u}{1-u}\right) & \text{for } t = 0, \\ \frac{1}{c_2(t)}\ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \text{for } t > 0. \end{cases}$$

The moment-generating function also depends on t and is given by

$$\mathcal{M}_{GSH}(u;0,1,t) = \begin{cases} \frac{\pi}{t}\sin(ut/c_2(t))\csc(u\pi/c_2(t)) & \text{for } t \in (-\pi, 0), \\ \sqrt{3}u\csc(\sqrt{3}u) & \text{for } t = 0, \\ \frac{\pi}{t}\sinh(ut/c_2(t))\csc(u\pi/c_2(t)) & \text{for } t > 0. \end{cases}$$

Further, the score function is given by

$$S(x) = -\frac{f'(x)}{f(x)} = \frac{c_2(t)\;(\exp(2\,c_2(t)\,x) - 1)}{\exp(2\,c_2(t)\,x) + 2\,a(t)\exp(c_2(t)\,x) + 1}$$

5 Derivation of the half-moments of the GSH distribution

In order to introduce a skew version of the GSH distribution, we have to calculate the (positive) half moments $\mathbf{E}^+(X^{2n+1}) = \int_0^\infty x^{2n+1} f(x)dx$ of the GSH distribution. The following result seems to be complicated at first sight, but the series involved are rapidly converging.

Proposition 5.1 (Half moments of the GSH distribution) *Assume X follows a GSH variable with density given in (4.5). Set $c = c_2(t), \overline{c} = c_1(t)$ and $a = a(t)$. Then*

$$\frac{\mathbf{E}^+(X^{2n+1})}{\overline{c} \cdot (2n+1)!} = \begin{cases} -\frac{\Gamma(2n+2)}{c^{2n+2}} \frac{\csc(t)}{(2n+1)!} \sum_{k=1}^{\infty} (-1)^k \cdot \frac{\sin(kt)}{k^{2n+2}}, & -\pi < t \leq 0 \\ \frac{\operatorname{csch}(t)}{c^{2n+2}} \left[\frac{t^{2n+2}}{2(2n+2)!} + \sum_{m=0}^{n} f_1(m) + \sum_{j=1}^{\infty} f_2(j) \right], & t > 0 \end{cases}$$

with

$$f_1(m) = \frac{t^{2m}}{(2m)!} \left(1 - 2^{2m-2n-1}\right) \zeta(2n+2-2m),$$

$$f_2(j) = (-1)^j j^{-2n-2} \exp(-jt).$$

Here ζ denotes the well known Riemann zeta function.

Proof: Except for a multiplicative constant, the problem is to find efficient formulae for evaluating

$$\int_0^\infty \frac{x^{2n+1} \exp(cx)}{\exp(2cx) + 2a \exp(cx) + 1} dx$$

where $n \in \mathbb{N} \cup \{0\}$, c is a positive constant, and $a = \cos(t)$ when $-\pi < t \leq 0$ and $a = \cosh(t)$ when $t > 0$. By using symmetry, the even moments can be computed from the original formulae outlined in Vaughan (2002).

Case 1: Assume $-\pi < t \leq 0$ and $a = \cos(t)$. Then

$$\int_0^\infty \frac{x^{2n+1}\exp(cx)dx}{\exp(2cx)+2\cos(t)\exp(cx)+1} = \frac{1}{c^{2n+2}}\int_1^\infty \frac{(\ln(v))^{2n+1}}{v^2+2v\cos(t)+1}dv$$

$$= -\frac{1}{c^{2n+2}}\int_0^1 \frac{(\ln(v))^{2n+1}}{v^2+2v\cos(t)+1}dv = -\frac{\Gamma(2n+2)}{c^{2n+2}}\csc(t)\sum_{k=1}^\infty (-1)^k \frac{\sin(kt)}{k^{2n+2}}.$$

Here we use the fact that for the Generalized Secant Hyperbolic distribution, the odd moments are 0. The right side of the last line follows from Gradshteyn and Ryhzik (1994, formula 4.272.3).

Case 2: Assume $t > 0$ and $a = \cosh(t)$ and define for $0 < \mu < 2$,

$$F(\mu) = \int_1^\infty \frac{u^{\mu-1}}{u^2+2u\cosh(t)+1}du$$

.

(a) First, let $\mu < 1$. Then define

$$\begin{aligned} F(\mu) &= \int_1^\infty \frac{u^{\mu-1}}{u^2+2u\cosh(t)+1}du \\ &= \frac{\text{csch(t)}}{2}\left[\int_1^\infty \frac{u^{\mu-1}}{u+\exp(-t)}du - \int_1^\infty \frac{u^{\mu-1}}{u+\exp(t)}du\right] \\ &= \frac{\text{csch(t)}}{2}\left[\frac{1}{1-\mu} + \sum_{k=1}^\infty \frac{1}{k+1-\mu}(-1)^k\exp(-kt)) - \frac{\exp((\mu-1)t)\pi}{\sin(\pi\mu)}\right. \\ &\qquad \left. + \sum_{k=0}^\infty \frac{(-1)^k\exp(-(k+1)t)}{k+\mu}\right]. \end{aligned}$$

These follow from Gradshteyn and Ryzhik(1994, 3.194.1, 3.194.2 and 3.194.3) and the fact that $\Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin(\pi\mu)}$ from Gradshteyn and Ryzhik (1994, 8.334.3).

Define

$$G(u) = \frac{\pi}{u}\left(1 - \frac{u\exp(-(t/\pi)u)}{\sin(u)}\right),$$

for $-\pi < u < 0$ or $0 < u < \pi$ and $G(0) = t$. This function is analytic on $(-\pi, \pi)$.

Using the product of two series (see, e.g., Gradshteyn and Ryzhik (1994, 0.316) yields:

$$G(u) = \frac{\pi}{u}\left(1 - \sum_{j=0}^{\infty}\frac{(-1)^j(t/\pi)^j u^j}{j!}\sum_{k=0}^{\infty} b_k u^k\right) = \frac{\pi}{u}\left(1 - \sum_{k=0}^{\infty} C_k u^k\right)$$

where $b_0 = 1$, $b_{2j-1} = 0$, $b_{2j} = 2(1-2^{1-2j})\pi^{-2j}\zeta(2j),\ j \in \mathbb{N}$, with $C_0 = 1$ and otherwise

$$C_k = \sum_{j=0}^{k}\frac{(-1)^j(t/\pi)^j}{j!}b_{k-j}.$$

Thus

$$G(u) = -\pi\sum_{k=1}^{\infty} C_k u^{k-1}.$$

Set

$$H(y) = G(\pi y) = -\sum_{k=1}^{\infty} C_k \pi^k y^{k-1}$$

and then

$$H^{(m)}(0) = -m!\sum_{k=0}^{m+1}\frac{(-1)^k(t/\pi)k}{k!}b_{m+1-k}\pi^{m+1}$$

from which we have for $m = 2n+1$:

$$H^{(2n+1)}(0) = -(2n+1)!\left(\frac{t^{2n+2}}{(2n+2)!} - \sum_{k=0}^{n}\frac{t^{2k}}{(2k)!}b_{2n+2-2k}\pi^{2n+2-2k}\right).$$

(b) If $\mu = 1$:

$$F(1) = \int_1^\infty \frac{dv}{(v+\exp(t))(v+\exp(-t))}$$

$$= \lim_{n\to\infty}\frac{\operatorname{csch}(t)}{2}\Big[\ln(v+\exp(-t)) - \ln(v+\exp(t))\Big]_1^n$$

$$= \frac{\operatorname{csch}(t)}{2}\lim_{n\to\infty}\left[\ln\left(\frac{n+\exp(-t)}{n+\exp(t)}\right) + \ln\left(\frac{1+\exp(-t)}{1+\exp(t)}\right)\right]$$

$$= \frac{t\cdot\operatorname{csch}(t)}{2} = \lim_{\mu\to 1^-}\int_1^\infty \frac{v^{\mu-1}}{(v+\exp(t))(v+\exp(-t))}dv.$$

(c) If $2 > \mu > 1$ then note first that

$$\frac{v^{\mu-1}}{(v+\exp(t))(v+\exp(-t))} = \frac{\operatorname{csch}(t)}{2}\left[\exp(t)\frac{v^{\mu-2}}{v+\exp(t)} - \exp(-t)\frac{v^{\mu-2}}{v+\exp(-t)}\right].$$

Using Gradshteyn and Ryzhik (1994, 3.194.1, 3.194.2 and 3.194.3) with $\alpha = \mu - 1$ and rearranging terms shows that again in this case,

$$F(\mu) = \frac{\operatorname{csch(t)}}{2}\left[H(1-\mu) + \sum_{k=1}^{\infty}\frac{1}{k+1-\mu}(-1)^k\exp(-kt)) + \sum_{k=0}^{\infty}\frac{(-1)^k\exp(-(k+1)t)}{k+\mu}\right].$$

Further, $\lim_{\mu\to 1^+} F(\mu) = t\operatorname{csch}(t)/2$, so that F is continuous at 1.

Finally, $F(\mu)$ has derivatives (with respect to μ in an interval containing $\mu = 1$) of arbitrary order, and hence

$$\begin{aligned}
F^{(2n+1)}(1) &= \int_1^\infty \frac{(\ln u)^{2n+1}}{u^2 + 2u\cosh(t) + 1}du \\
&= \frac{\operatorname{csch(t)}}{2}\left[-H^{(2n+1)}(0) + (2n+1)!\sum_{k=1}^{\infty}\frac{(-1)^k\exp(-kt)}{k^{2n+2}}\right. \\
&\quad \left. +(2n+1)!\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\exp(-(k+1)t)}{(k+1)^{2n+2}}\right] \\
&= \operatorname{csch}(t)(2n+1)!\left[\frac{t^{2n+2}}{2(2n+2)!} + \sum_{m=0}^{n}\frac{t^{2m}}{(2m)!}(1-2^{2m-2n-1})\zeta(2n+2-2m)\right. \\
&\quad \left. +\sum_{j=1}^{\infty}(-1)^j j^{-2n-2}\exp(-jt)\right].
\end{aligned}$$

which involves the well known Riemann zeta function and rapidly converging series.

□

6 A class of skewed GSH distributions

6.1 Density and (inverse) cumulative distribution function

There are plenty of methods in the literature to make a symmetric distribution skewed. As the distribution function of the GSH distribution is explicitly known, we lay our focus on splitting the scale parameter, as it was done by Fernandez, Osiewalski and Steel (1995) for the skewed exponential power distribution.

Let $\gamma > 1$, $\mathbf{I}^{+}(x)$ denote the indicator function for x on $\mathbb{R}_{+}$ and $\mathbf{I}^{-}(x)$ denote the indicator function for x on $\mathbb{R}_{-}$. Then it can be easily verified that

$$f_{SGSH}(x;t,\gamma) = \frac{2}{\gamma+\frac{1}{\gamma}}\left\{f_{GSH}(x/\gamma)\cdot\mathbf{I}^{-}(x) + f_{GSH}(\gamma x)\cdot\mathbf{I}^{+}(x)\right\}$$

$$= \frac{2c_1}{\gamma+\frac{1}{\gamma}}\cdot\left(\frac{\exp(c_2 x/\gamma)\cdot\mathbf{I}^{-}(x)}{\exp(2c_2 x/\gamma)+2a\exp(c_2 x/\gamma)+1} + \frac{\exp(c_2\gamma x)\cdot\mathbf{I}^{+}(x)}{\exp(2c_2\gamma x)+2a\exp(c_2\gamma x)+1}\right)$$

is a density function which is symmetric for $\gamma = 1$, skewed to the right for $\gamma > 1$ and skewed to the left for $0 < \gamma < 1$. The corresponding distribution will be termed as *skewed generalized secant hyperbolic distribution* in the sequel. The effect of the skewness parameter on the density can be seen in figure 1, below.

It can also be shown (see, for example, Grottke (2002), p. 21) that the cumulative and the inverse cumulative distribution functions admits a closed form, namely

$$F_{SGSH}(x;t,\gamma) = \frac{2\gamma^2}{\gamma^2+1}\cdot\left(F_{GSH}\left(x/\gamma\right)\cdot\mathbf{I}^{-}(x) + \left(\frac{\gamma^2-1+2F_{GSH}(\gamma x)}{2\gamma^2}\right)\cdot\mathbf{I}^{+}(x)\right),$$

$$F^{-1}_{SGSH}(x;t,\gamma) = \gamma F^{-1}_{GSH}\left(x\cdot\frac{\gamma^2+1}{2\gamma^2}\right)\mathbf{I}^{A}(x) + \frac{1}{\gamma}F^{-1}_{GSH}\left(x\cdot\frac{\gamma^2+1}{2}-\frac{\gamma^2-1}{2}\right)\mathbf{I}^{\overline{A}}(x).$$

with

$$\mathbf{I}^{A}(x) = \begin{cases} 1, & \text{if } x < \frac{\gamma^2}{1+\gamma}, \\ 0, & \text{if } x \geq \frac{\gamma^2}{1+\gamma}. \end{cases} \qquad \text{and} \qquad \mathbf{I}^{\overline{A}}(x) = 1 - \mathbf{I}^{A}(x).$$

6.2 Moments of the SGSH distribution

Using proposition 5.1, we can deduce the moments of a SGSH distribution, in particular calculating the skewness and kurtosis coefficient, measured by the third and fourth standardized moments. The proceeding is as follows: Assume X follows a SGHS distribution and let X_1 denote the corresponding symmetric (i.e. with $\gamma = 1$) random variable with density f_1. Then

$$\mathbf{E}(X^r) = \mathbf{E}^+(X_1^r) \cdot \frac{2\gamma}{\gamma^2+1} \cdot \left[(-1)^r \gamma^{r+1} + \gamma^{-r-1}\right], \tag{6.6}$$

with positive half moments

$$\mathbf{E}^+(X_1^r) = \int_0^\infty x^r f_1(x) dx.$$

Note, that *for odd* r we can approximate the positive half moments by applying proposition 5.1. For simplicity, let. *For even* r, the positive half moments can be obtained from $\mathbf{E}(X_1^r)$ by division with 2. In the case of a standard GSH variable X_1, i.e. with $\mathbf{E}(X_1) = 0$ and $\mathbf{E}(X_1^2) = \mathbf{V}(X_1) = 1$, Vaughan (2002) showed that

$$\mathbb{K}(X_1) = \mathbf{E}(X_1^4) = \begin{cases} \frac{21\pi^2 - 9t^2}{5\pi^2 - 5t^2}, & \text{for } t \in (-\pi, 0] \\ \frac{21\pi^2 + 9t^2}{5\pi^2 + 5t^2}, & \text{for } t > 0. \end{cases}$$

With this results in mind, it is straightforward to deduce the power moments and the central moments of a standard SGSH variable X. The results are summarized in the lemma 6.1 and lemma 6.2, below.

Lemma 6.1 (Power moments) *Assume X follows a standard SGSH variable with parameter $t > -\pi/2$ and $\gamma > 0$. Then the second and fourth power moments $m_i' = \mathbf{E}(X^i)$, $i = 2, 4$ are given by*

$$m_2' = \frac{\gamma^6 + 1}{\gamma^2(\gamma^2 + 1)}$$

$$m'_4 = \begin{cases} \frac{21\pi^2-9t^2}{5\pi^2-5t^2} \cdot \frac{\gamma^{10}+1}{(\gamma^2+1)\gamma^4}, & \textit{for } t \in (-\pi, 0], \\ \frac{21\pi^2+9t^2}{5\pi^2+5t^2} \cdot \frac{\gamma^{10}+1}{(\gamma^2+1)\gamma^4}, & \textit{for } t > 0. \end{cases}$$

Lemma 6.2 (Central moments) *Assume X follows a standard SGSH variable with $t > -\pi/2$ and $\lambda > 0$. Then*

$$m_2 = \mathbf{E}(X - m'_1)^2 = \mathbf{V}(X) = \frac{\gamma^6 + 1}{\gamma^2(\gamma^2+1)} - (m'_1)^2,$$

$$m_3 = \mathbf{E}(X - m'_1)^3 = m'_3 - 3\frac{\gamma^6 + 1}{\gamma^2(\gamma^2+1)}m'_1 + 2(m'_1)^3,$$

$$m_4 = \mathbf{E}(X - m'_1)^4 = m'_4 - 4m'_3(m'_1) + 6\frac{\gamma^6 + 1}{\gamma^2(\gamma^2+1)}(m'_1)^2 - 3(m'_1)^4.$$

Finally, using lemma 6.2, the skewness coefficient $\mathbb{S}(X)$ and the kurtosis coefficient $\mathbb{K}(X)$ can easily be deduced. The results are summarized in table 1 and 2, below.

Table 1: Influence of t and λ on $\mathbb{S}(X)$.

$\lambda \downarrow, t \rightarrow$	0.50	0.90	0.95	1.00	1.05	1.10	1.50	2.00
-3.0	4.8520	1.4510	0.7210	0.0000	-0.6870	-1.3190	-4.1360	-4.8520
-2.0	1.7550	0.4140	0.2040	0.0000	-0.1940	-0.3750	-1.3330	-1.7550
-1.0	1.3530	0.3020	0.1480	0.0000	-0.1410	-0.2740	-0.9980	-1.3530
-0.5	1.2820	0.2830	0.1390	0.0000	-0.1320	-0.2570	-0.9400	-1.2820
0.0	1.2570	0.2780	0.1370	0.0000	-0.1300	-0.2520	-0.9230	-1.2570
0.5	1.2390	0.2720	0.1340	0.0000	-0.1270	-0.2470	-0.9060	-1.2390
1.0	1.1810	0.2570	0.1260	0.0000	-0.1200	-0.2330	-0.8590	-1.1810
2.0	0.9960	0.2100	0.1030	0.0000	-0.0980	-0.1910	-0.7130	-0.9960
20.0	0.0490	0.0090	0.0040	0.0000	-0.0040	-0.0080	-0.0320	-0.0490

Table 2: Influence of t and λ on $\mathbb{K}(X)$.

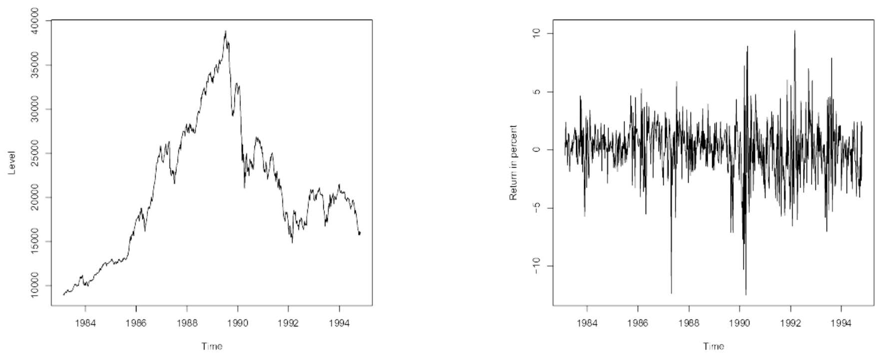
$\lambda \downarrow, t \rightarrow$	0.50	0.90	0.95	1.00	1.05	1.10	1.50	2.00
-3.0	42.4610	30.2350	29.3340	29.0390	29.3060	30.0270	38.9620	42.4610
-2.0	8.3710	5.9670	5.8670	5.8360	5.8640	5.9430	7.2440	8.3710
-1.0	6.1240	4.5470	4.4890	4.4710	4.4870	4.5330	5.3340	6.1240
-0.5	5.7780	4.3310	4.2790	4.2620	4.2770	4.3190	5.0450	5.7780
0.0	5.6580	4.2660	4.2160	4.2000	4.2140	4.2540	4.9520	5.6580
0.5	5.5750	4.2050	4.1560	4.1410	4.1550	4.1930	4.8760	5.5750
1.0	5.3050	4.0370	3.9930	3.9790	3.9920	4.0270	4.6520	5.3050
2.0	4.5180	3.5500	3.5180	3.5080	3.5170	3.5420	4.0040	4.5180
20.0	1.8800	1.8580	1.8580	1.8580	1.8580	1.8580	1.8670	1.8800

It can be seen that both skewness and kurtosis are affected by t and γ and that there is no separation between skewness and kurtosis by t and γ. However, introducing the parameter γ leads to a very flexible distribution family. This will be demonstrated in the context of financial return data in the next section.

7 Application with respect to financial return data

In order to adopt and compare estimation results for a great deal of distributions – in particular the stable distributions – priority is given to the weekly returns of the Nikkei from July 31, 1983 to April 9, 1995, with $N = 608$ observations. This series was intensively investigated, for example, by Mittnik, Paolella and Rachev (1998) because it exhibits typical stylized facts of financial return data. Figure 1 illustrates the time series of levels and corresponding log-returns.

Figure 1: Levels and returns of Nikkei.



(a) Levels (b) Returns

Similar to Mittnik, Paolella and Rachev (1998), four criteria are employed to compare the goodness-of-fit of the different candidate distributions. The first is the *log-Likelihood value* ($\mathcal{LL}$) obtained from the Maximum-Likelihood estimation. The $\mathcal{LL}$-value can be considered as an "overall measure of goodness-of-fit and allows us to judge which candidate is more likely to have generated the data". As distributions with different numbers of parameters k are used, this is taken into account by calculating the *Akaike criterion* given by

$$AIC = -2 \cdot \mathcal{LL} + \frac{2N(k+1)}{N-k-2}.$$

The third criterion is the *Kolmogorov-Smirnov distance* as a measure of the distance between the estimated parametric cumulative distribution function, $\hat{F}$, and the empirical sample distribution, F_{emp}. It is usually defined by

$$\mathcal{K} = 100 \cdot \sup_{x \in \mathbb{R}} |F_{emp}(x) - \hat{F}(x)|. \tag{7.7}$$

Finally, *Anderson-Darling statistic* is calculated, which weights $|F_{emp}(x) - \hat{F}(x)|$ by the reciprocal of the standard deviation of F_{emp}, namely $\sqrt{\hat{F}(x)(1-\hat{F}(x))}$, that is

$$\mathcal{AD}_0 = \sup_{x\in\mathbb{R}} \frac{|F_{emp}(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1-\hat{F}(x))}}. \tag{7.8}$$

Instead of just the maximum discrepancy, the second and third largest value, which is commonly termed as $\mathcal{AD}_1$ and $\mathcal{AD}_2$, are also taken into consideration. Whereas $\mathcal{K}$ emphasizes deviations around the median of the fitted distribution, $\mathcal{AD}_0, \mathcal{AD}_1$ and $\mathcal{AD}_2$ allow discrepancies in the tails of the distribution to be appropriately weighted.

Estimation was performed not only for the two families of generalized hyperbolic secant distributions, but also for distribution families which have became popular in finance in the last years: Firstly, the generalized hyperbolic (GH) distributions which were discussed by Prause (1999) and include, for example, the Normal-inverse Gaussian (NIG) distributions (see Barndorff-Nielsen (1997)) as well as the hyperbolic (HYP) distributions (see Eberlein and Keller (1995)) as special cases. Secondly, the exponential generalized beta of the second kind (EGB2) distribution that was introduced by McDonald (1991) as a generalization of the logistic (LOG) distribution and used in various financial applications, see also Fischer (2002). Thirdly, a very flexible generalization of the generalized t-distribution (SGT2) proposed by Grottke (2001). Finally, we performed calculations for the gh-transformed normal (gh-NORM) distribution (see Klein and Fischer (2002)). The estimation results are summarized in table 3, below. They have been obtained using a Matlab routine which was written by the authors.

Table 3: Goodness-of-fit: Nikkei225.

Distr.	**k**	$\mathcal{LL}$	**AIC**	$\mathcal{K}$	$\mathcal{AD}_\mathbf{0}$	$\mathcal{AD}_\mathbf{1}$	$\mathcal{AD}_\mathbf{2}$
NORM	2	-1428.3	2862.6	6.89	4.920	2.810	1.070
STABLE	4	-1393.2	2796.5	3.00	0.085	0.084	0.081
HS	2	-1393.4	2792.8	4.31	0.216	0.150	0.121
GHS	3	-1392.2	2794.6	4.15	0.140	0.117	0.114
SGHS	4	-1388.1	2786.3	2.42	0.091	0.090	0.083
GSH	3	-1392.3	2794.8	4.17	0.142	0.117	0.114
SGSH	4	-1387.5	2785.2	2.18	0.088	0.087	0.080
LOG	2	-1398.1	2802.1	4.56	0.362	0.236	0.186
EGB2	4	-1388.1	2786.3	2.45	0.103	0.100	0.095
GH	5	-1388.0	2788.2	2.43	0.095	0.093	0.086
HYP	4	-1388.2	2786.5	2.50	0.106	0.103	0.098
NIG	4	-1388.2	2786.6	2.48	0.085	0.085	0.075
SGT2	5	-1387.4	2786.9	2.12	0.076	0.072	0.071
Student-t	3	-1392.2	2792.5	3.77	0.107	0.104	0.103
gh-NORM	4	-1388.7	2788.5	2.27	0.068	0.062	0.061

Firstly, let us focus on the fit of generalized hyperbolic secant families. There seems to be no difference between the GSH distribution of Vaughan (2002) and the GHS distribution of Harkness and Harkness (1968). This is not true if we consider the skewed pendants and compare the NEF-GHS distribution of Morris (1982) with the SGSH distribution (which was introduced in the last section) which exhibits better values with respect to all five criteria. For that reason, we restrict our considerations to SGHS distribution. Concerning the $\mathcal{LL}$-value, only the SGT2 distribution has a higher value. The same is true if we compare the $\mathcal{K}$-values. If we take the number of parameters into account (i.e. focus on the *AIC* criterion), SGSH even outperforms SGT2. The situation is a little bit different for the fit in the tails. Here, gh-transformed distributions finished best, followed by SGT2, NIG, STABLE and SGHS (Note, that the last three are close together).

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