The Folded EGB2 distribution and its application to financial return data

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Abstract

In the literature there are several generalizations of the standard logistic distribution. Most of them are included in the generalized logistic distribution of type IV or EGB2 distribution. However, this four parameter family fails in modeling skewness absolutly greater than 2 and kurtosis higher than 9. To remove this shortcoming, an additional parameter is introduced. Unfortunately, there is now no closed form for the probability density function of the generalized EGB2, briefly called FEGB2 or generalized logistic distribution of type V. However, it can be approximated numerically, for example by saddlepoint approximation or numerical integration methods. Finally, FEGB2 is used for modeling returns of financial data.

1 Introduction

The aim of this paper is to propose a generalization of the EGB2 distribution or generalized logistic distribution of type IV. The EGB2 distribution itsself is a four parameter generalization of the standard logistic distribution. It can be used to model data which are skewed and leptokurtic. Section 2 gives a short review of the EGB2 distribution including the special cases. However, only a restricted domain of skewness and leptocurtosis can be achieved by EGB2 if skewness and kurtosis are measured by the third and fourth standardized moments. To remove this shortcoming, an additional parameter, the so-called convolution parameter, is introduced in section 3. Unfortunately, there is now no closed form for the probability density function of the generalized EGB2, briefly called FEGB2. However, it can be approximated numerically for example by saddlepoint approximation or numerical integration methods. This is demonstrated in section 4. Finally, the generalized EGB2 is applied to financial return data in section 5.

2 Generalizing the logistic distribution: A review

2.1 Generalized logistic distributions of type I, II and III

In the literature there are several generalizations of the logistic distribution. Following the notation of Johnson, Kotz & Balakrishnan [16] the type I generalized logistic distribution (GL_I) is characterized by its cumulative distribution function

$$F(x;\alpha) = \frac{1}{(1+e^{-x})^{\alpha}}, \qquad \alpha > 0, x \in \mathbb{R} \qquad \Longrightarrow \qquad f(x;\alpha) = \frac{\alpha e^{-x}}{(1+e^{-x})^{\alpha+1}}.$$

Hereby, α determines the skewness of the distribution, namely GL_I is negatively skewed for $0 < \alpha < 1$, positively skewed for $\alpha > 1$ and symmetric for $\alpha = 1$. In addition, Zelterman [30],[31] introduced a scale parameter μ and a location parameter δ.

Another generalization of the logistic distribution is given by the type II generalized logistic distribution (GL_{II}) with cumulative distribution function

$$F(x;\alpha) = 1 - \frac{e^{-\alpha x}}{(1+e^{-x})^{\alpha}}, \qquad \alpha > 0, x \in \mathbb{R} \qquad \Longrightarrow \qquad f(x;\alpha) = \frac{\alpha e^{-\alpha x}}{(1+e^{-x})^{\alpha+1}}.$$

It can be shown that if $X \sim \mathrm{GL}_I$, then $-X \sim \mathrm{GL}_{II}$. Therefore GL_{II} is sometimes also called negative GL_I. Consequently the GL_{II} is positively skewed for $0 < \alpha < 1$ and negatively skewed for $\alpha > 1$.

Finally, Davidson [8] used the type III generalized logistic distribution (GL_{III}) (which bears no relationship to GL_I except the special case of the logistic distribution) having symmetric probability density function

$$f(x;\beta) = \frac{1}{B(\beta,\beta)} \cdot \frac{\exp(-\beta x)}{[1+\exp(-x)]^{2\beta}}, \qquad \beta > 0, \quad x \in \mathbb{R}.$$

Here $B(a,b)$ denotes the Beta function, whose definition and main properties are summarized in Appendix A. Obviously, setting $\beta = 1$ yields the standard logistic density function. Moreover, it can be shown that GL_{III} has thicker tails than the normal distribution. George and Ojo [13] and George, El-Saidi and Singh [12] developed an approximation to Student's t distribution with ν degrees of freedom based on GL_{III}. In order to match the coefficient of kurtosis they recommend the use of $\beta = \frac{\nu - 3.25}{5.5}$.

2.2 The EGB2 or generalized logistic distribution of type IV

1. Definition: It is readily observed, that GL_I (setting $\beta_1 = 1, \beta_2 = \alpha$) and GL_{III} (setting $\beta_1 = \beta_2$) are included in the *type IV generalized logistic distribution* with density function

$$f_-(x;\beta_1,\beta_2) = \frac{1}{B(\beta_1,\beta_2)} \cdot \frac{\exp(-\beta_1 x)}{[1+\exp(-x)]^{\beta_1+\beta_2}}, \quad x \in \mathbb{R}. \tag{2.1}$$

GL_{IV} is sometimes also referred to as the *exponential generalized beta of the second kind*, denoted by EGB2 (see McDonald [19]), or as *z-distribution* (see Barndorff-Nielsen, Kent, Soerensen [3]). We will follow the notation of McDonald and term this distributions as EGB2 in the sequel.

There is also another definition of EGB2 (see also McDonald [19]), namely

$$f_+(x;\beta_1,\beta_2) = \frac{1}{B(\beta_1,\beta_2)} \cdot \frac{\exp(\beta_1 x)}{[1+\exp(x)]^{\beta_1+\beta_2}}, \quad x \in \mathbb{R}. \tag{2.2}$$

by the end of this paper definition (2.2) will be used to deduce further results. The connection between f_+ and f_- is stated in the following Lemma.

Lemma 2.1 *Let f_+ and f_- be defined as in (2.2) and (2.1). Then*

$$f_+(x,\beta_1,\beta_2) = f_-(x,\beta_2,\beta_1).$$

Proof:

$$\begin{aligned} f_+(x,\beta_1,\beta_2) &= \frac{1}{B(\beta_1,\beta_2)} \cdot \frac{(e^x)^{\beta_1}}{(1+e^x)^{\beta_1}(1+e^x)^{\beta_2}} \\ &= \frac{1}{B(\beta_2,\beta_1)} \cdot \left(\frac{e^x}{1+e^x}\right)^{\beta_1} \cdot \left(\frac{1}{1+e^x}\right)^{\beta_2} \\ &= \frac{1}{B(\beta_2,\beta_1)} \cdot \left(\frac{1}{e^{-x}+1}\right)^{\beta_1} \cdot \left(\frac{e^{-x}}{e^{-x}+1}\right)^{\beta_2} \\ &= \frac{1}{B(\beta_2,\beta_1)} \cdot \frac{(e^{-x})^{\beta_2}}{(1+e^{-x})^{\beta_1}(1+e^{-x})^{\beta_2}} = f_-(x,\beta_2,\beta_1) \qquad \square \end{aligned}$$

Finally, following Zelterman and Balakrishnan [32] (2.1) can alternatively be written as

$$f(x;\beta_1,\beta_2) = \frac{1}{B(\beta_1,\beta_2)} \cdot [F^*(x)]^{\beta_1}[1-F^*(x)]^{\beta_2}, \qquad x \in \mathbb{R}, \tag{2.3}$$

where F^* denotes is the standard logistic cumulative distribution function.

The positive parameter β_1 and β_2 determine the skewness in the following manner:

$$\text{For} \left\{ \begin{array}{c} \beta_1 > \beta_2 \\ \beta_1 < \beta_2 \\ \beta_1 = \beta_2 \end{array} \right\} \text{ the distribution is } \left\{ \begin{array}{l} \text{positively skewed} \\ \text{negatively skewed} \\ \text{symmetric} \end{array} \right\}.$$

Introducing a location parameter μ and a scale parameter δ leads to a four parameter family with probability density function

$$f(x; \mu, \delta, \beta_1, \beta_2) = \frac{1}{\delta B(\beta_1, \beta_2)} \cdot \frac{\exp(\beta_1 \frac{x-\mu}{\delta})}{[1 + \exp(\frac{x-\mu}{\delta})]^{\beta_1+\beta_2}}, \quad x \in \mathbb{R}. \tag{2.4}$$

2. History and applications: The EGB2 appeared 1921 for the first time in the work of Fisher [11]. Meanwhile there are plenty of applications of that distribution family: Prentice [27] proposed type IV as an alternative for modelling binary response data to the usual logistic model. McDonald/White [22] used it in the context of modelling the error-distribution for regression models, McDonald and Nelson [21] in the special case of beta estimation in the market model, McDonald/Xu [23] as error-distribution for ARIMA models and Tiku/Wong/Bian [29] in the context of modeling time series with asymmetric innovations.

3. Mixture representations: In the literature there exists several mixture representations of the EGB2 distribution. Firstly, (2.4) can represented in terms of the generalized beta of the second kind (GB2) (see McDonald [19])

$$f_{EGB2}(x; \mu, \delta, \beta_1, \beta_2) = f_{GB2}(e^x; \delta^{-1}, e^\mu, \beta_1, \beta_2) \cdot e^x,$$

where the definition of the generalized beta of the second kind distribution can be found in appendix B. Thus, X distributed according to EGB2 implies that e^X is distributed as GB2, or if X is distributed as GB2, then $\ln(X)$ is distributed according to EGB2. Therefore EGB2 is also termed as the LGB2 or lnGB2. Bookstaber and McDonald [6] demonstrate the value of modeling security market returns with GB2.

Secondly, an extension of this result can be found in McDonald and Butler [20] who show that the EGB2 can be interpreted as an exponential generalized gamma distribution (EGG), which has random scale parameter whose distribution is an inverse generalized gamma distribution (IGG) (Both definitions are stated in appendix B):

$$f_{EGB2}(x; \mu, \delta, \beta_1, \beta_2) = \int_0^\infty f_{EGG}(x; 1/\delta, \theta, \beta_1) f_{IGG}(\theta; 1/\delta, e^\mu, \beta_2) d\theta.$$

Thirdly, another similar mixture representation in terms of the generalized gamma distribution is given by McDonald [19]:

$$f_{EGB2}(x; \mu, \delta, \beta_1, \beta_2) = \int_0^\infty f_{GG}(e^x; 1/\delta, \theta, \beta_1)\, e^x\, f_{GG}(\theta; -1/\delta, \delta, \beta_2) d\theta.$$

4. Moment generating function, moments, skewness and kurtosis: It is easily seen that the moment generating function of $X \sim EGB2(\mu, \delta, \beta_1, \beta_2)$ is

$$\mathcal{M}_X(t) = \exp(\mu\, t) \cdot \frac{B(\beta_1 + \delta t, \beta_2 - \delta t)}{B(\beta_1, \beta_2)}, \qquad -\frac{\beta_1}{\delta} < t < \frac{\beta_2}{\delta}, \tag{2.5}$$

which leads to the characteristic function

$$\varphi(t) = \mathcal{M}(it) = e^{\mu\, i\, t} \cdot \frac{B(\beta_1 + i\delta t, \beta_2 - i\delta t)}{B(\beta_1, \beta_2)} = e^{\mu\, i\, t} \cdot \frac{\Gamma(\beta_1 + i\delta t)\Gamma(\beta_2 - i\delta t)}{\Gamma(\beta_1)\Gamma(\beta_2)}.$$

From (2.5) we get (see also McDonald [19]) and appendix A for the definition of the Psi function ψ):

$$E(X) = \delta[\psi(\beta_1) - \psi(\beta_2)] + \mu, \tag{2.6}$$

$$M_2 = Var(X) = \delta^2[\psi'(\beta_1) + \psi'(\beta_2)], \tag{2.7}$$

$$M_3 = E[(X-\mu)^3] = \delta^3[\psi''(\beta_1) - \psi''(\beta_2)], \tag{2.8}$$

$$M_4 = E[(X-\mu)^4] = \delta^4\{\psi'''(\beta_1) + \psi'''(\beta_2) + 3[\psi'(\beta_1) + \psi'(\beta_2)]^2\}. \tag{2.9}$$

Hence, one calculate the statistics for skewness and kurtosis

$$\mathbb{S}(X) = \frac{M_3}{M_2^{1.5}} = \frac{\psi''(\beta_1) - \psi''(\beta_2)}{\sqrt{\psi'(\beta_1) + \psi'(\beta_2)}^3}, \tag{2.10}$$

$$\mathbb{K}(X) = \frac{M_4}{M_2^2} = \frac{\psi'''(\beta_1) + \psi'''(\beta_2) + 3[\psi'(\beta_1) + \psi'(\beta_2)]^2}{[\psi'(\beta_1) + \psi'(\beta_2)]^2}. \tag{2.11}$$

It can be shown that the EGB2 can accomodate skewness values between -2 und 2. The summary of the entries in the following table reflect the role of the parameter β_1 und β_2.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	0.00	-1.97	-2.00	-2.00	-2.00	-2.00	-2.00	-2.00
0.10	1.97	0.00	-1.91	-1.94	-1.95	-1.96	-1.96	-1.96
1	2.00	1.91	0.00	-0.58	-0.92	-1.03	-1.12	-1.13
2	2.00	1.94	0.58	0.00	-0.44	-0.61	-0.74	-0.76
5	2.00	1.95	0.92	0.44	0.00	-0.20	-0.41	-0.44
10	2.00	1.96	1.03	0.61	0.20	0.00	-0.24	-0.28
50	2.00	1.96	1.12	0.74	0.41	0.24	0.00	-0.06
100	2.00	1.96	1.13	0.76	0.44	0.28	0.06	0.00

Table 1: Range of skewness.

Besides, EGB2 is able to model positive kurtosis. The following table will demonstrate the flexibility of modeling leptocurtosis, which ranges between 3 and 9.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	6.00	8.88	9.00	9.00	9.00	9.00	9.00	9.00
0.10	8.88	5.92	8.65	8.76	8.81	8.82	8.83	8.83
1	9.00	8.65	4.20	4.33	4.87	5.12	5.34	5.37
2	9.00	8.76	4.33	3.59	3.69	3.88	4.12	4.15
5	9.00	8.81	4.87	3.69	3.22	3.22	3.37	3.40
10	9.00	8.82	5.12	3.88	3.22	3.10	3.15	3.17
50	9.00	8.83	5.34	4.12	3.37	3.15	3.02	3.02
100	9.00	8.83	5.37	4.15	3.40	3.17	3.02	3.01

Table 2: Range of kurtosis.

5. Log-density and ψ-function: In order to implement for example GARCH models it is helpful to calculate the log-density of the EGB2 distribution. Taking the logarithm of (2.4)

$$\ln(f(x;\mu,\delta,\beta_1,\beta_2)) = -\ln(\delta B(\beta_1,\beta_2)) + \beta_1\frac{x-\mu}{\delta} - (\beta_1+\beta_2)\ln\left(1+e^{\frac{x-\mu}{\delta}}\right). \quad (2.12)$$

Obviously, the density is asymptotically log-linear with asymptotics (setting for simplicity $\mu = 0$, $\delta = 1$ and $C = -\ln\{B(\beta_1,\beta_2)\}$)

$$A_1(x) = C - \beta_2 x \quad \text{and} \quad A_2(x) = C + \beta_1 x. \quad (2.13)$$

In addition, according to the work of McDonald and White [22] the ψ-function of the EGB2 is given by

$$\psi(u;\mu,\delta,\beta_1,\beta_2) = -\frac{f'(u;\mu,\delta,\beta_1,\beta_2)}{f(u;\mu,\delta,\beta_1,\beta_2)} = \frac{\beta_2 - \beta_1 e^{\frac{u-\mu}{\delta}}}{\delta(1+e^{\frac{u-\mu}{\delta}})} \quad (2.14)$$

and is bounded above by $\frac{\beta_2}{\delta}$ and below by $-\frac{\beta_1}{\delta}$. The bounds are asymmetric unless the underlying distribution is symmetric. If the EGB2 is skewed to the right ($\beta_1 > \beta_2$), the magnitude of the bound for positive values $\frac{\beta_2}{\delta}$ is greater than for negative values $-\frac{\beta_1}{\delta}$. Considering the symmetric case $\beta_1 = \beta_2$, Li and deMoor [17] showed that there is strong similarity in tails and the middle part between Huber's ψ-function and the ψ-function of EGB2.

6. Special cases and limiting cases. Farewell and Prentice [10] showed that EGB2 goes in limit to lognormal ($\beta_1 \rightarrow \infty$), to normal ($\beta_1 \rightarrow \infty, \beta_2 \rightarrow \infty$) and to Weibull ($\beta_1 = 1, \beta_2 \rightarrow \infty$). For $\beta_1 = \beta_2 = 1$ one gets the standard logistic distribution, EGB2($x; 0, \frac{1}{\sqrt{2\pi}}, \frac{1}{2}, \frac{1}{2}$) coincides with the hyperbolic cosine distribution. If X is beta distributed with parameter β_1 and β_2 then $\ln(\frac{X}{1-X}) \sim EGB2(x;0,1,\beta_1,\beta_2)$. The EGB2 also relates to the exponential generalized gamma (EGG) (see Cameron and White [7]) since

$$EGG(x;\mu,\delta,\beta_1) = \lim_{\beta_2\rightarrow\infty} EGB2(u;\mu+\delta\ln(\beta_2),\delta,\beta_1,\beta_2).$$

7. Self-decomposability and infinite divisibility: According to Barndorff-Nielsen, Kent and Soerensen [3], the EGB2 allows for the following normal variance-mean mixture representation[1]: Let $f_m(x;\delta,\gamma)$ be the probability density function of a random variable X on $\mathbb{R}_+$ which has moment generating function

$$\mathcal{M}(t) = \prod_{k=0}^{\infty}\left(1 - \frac{t}{\frac{1}{2}(\delta+k)^2 - \gamma}\right)^{-1}, \qquad \delta > 0, \gamma < \frac{1}{2}\delta^2,$$

i.e. $f_m(x;\delta,\gamma)$ lies in the class of infinite convolution of exponential distribution (Pólya distributions), then EGB2 is a normal variance-mean mixture with mixing distribution f_m. It was noted above that f_m is a infinite convolution of exponential distributions and hence belongs to the Thorin class for every (δ,γ). Results of Halgreen [15] and Thorin [28] imply that EGB2 belongs to the extended Thorin class (or class of generalized Gamma convolutions) and is therefore self-decomposable and hence infinitely divisible. Alternatively, Bondesson [5] proved directly that that the log-Gamma distribution and hence EGB2 belong to the extended Thorin class.

[1] Suppose X is a random variate which is (for a given u) Gauss distributed with mean $\mu + \beta u$ and variance u. Suppose moreover that u itsself follows a cumulative probability function F on $\mathbb{R}_+$. Then the distribution of X is said to be a **normal variance-mean mixture** with mixing distribution F. If $\beta = 0$ it is termed a normal variance mixture.

3 The folded EGB2 distribution (FEGB2)

1. Motivation: As we have seen in the previous section the EGB2 fails in modeling skewness higher than 2 or less than -2 and kurtosis higher than 9. In order to remove that shortcoming, let us introduce an additional parameter $\tau > 0$. In the following discussion we will concentrate for simplicity on the standard EGB2 distribution with $\mu = 0$ and $\delta = 1$.

2. Definition: Let φ_{EGB2} denote the characteristic function of an EGB2-distributed random variable X. As X is infinitely divisible, φ_{EGB2} to the power of τ ($\tau > 0$)

$$(\varphi_{EGB2}(t))^\tau = \left(\frac{B(\beta_1 + it, \beta_2 - it)}{B(\beta_1, \beta_2)}\right)^\tau = \left(\frac{\Gamma(\beta_1 + it)\Gamma(\beta_2 - it)}{\Gamma(\beta_1)\Gamma(\beta_2)}\right)^\tau \tag{3.15}$$

is again a characteristic function (see Lukacs [18]). The corresponding random variable X is said to be *FEGB2-distributed* or *generalized logistic of type V* with parameters β_1, β_2 and the so-called convolution parameter τ.

3. Moments, skewness and kurtosis:

Lemma 3.1 *Let $X \sim FEGB2(\beta_1, \beta_2, \tau)$ and $\mathcal{M}(u)$ denote the corresponding moment generating function. Define further $\mathcal{P}(u) = \tau[\psi(\beta_1 + u) - \psi(\beta_2 - u)]$, where ψ denotes again the Digamma function (see Appendix A for definition). Then*

$$\mathcal{M}'(u) = \mathcal{M}(u) \cdot \mathcal{P}(u). \tag{3.16}$$

Proof: As stated above, the moment generating function of X is given by

$$\mathcal{M}(u) = \left(\frac{B(\beta_1 + u, \beta_2 - u)}{B(\beta_1, \beta_2)}\right)^\tau.$$

Calculating the first derivative yields

$$\begin{aligned}\mathcal{M}'(u) &= \tau \cdot \left(\frac{B(\beta_1 + u, \beta_2 - u)}{B(\beta_1, \beta_2)}\right)^{\tau-1} \left(\frac{\Gamma'(\beta_1 + u)\Gamma(\beta_2 - u) - \Gamma(\beta_1 + u)\Gamma'(\beta_2 - u)}{\Gamma(\beta_1)\Gamma(\beta_2)}\right) \\ &= \left(\frac{B(\beta_1 + u, \beta_2 - u)}{B(\beta_1, \beta_2)}\right)^\tau \tau \left(\frac{\Gamma'(\beta_1 + u)}{\Gamma(\beta_1 + u)} - \frac{\Gamma'(\beta_2 - u)}{\Gamma(\beta_2 - u)}\right) = \mathcal{M}(u) \cdot \mathcal{P}(u). \quad \square\end{aligned}$$

Proposition 3.1 (Moments) *Let $X \sim FEGB2(\beta_1, \beta_2, \tau)$ and $M_i = E(X - m)^i$ with $m = E(X)$. Then*

$$m = E(X) = \tau\left(\psi(\beta_1) - \psi(\beta_2)\right), \tag{3.17}$$

$$M_2 = Var(X) = \tau \cdot \left(\psi'(\beta_1) + \psi'(\beta_2)\right), \tag{3.18}$$

$$M_3 = \tau \cdot \left(\psi''(\beta_1) - \psi''(\beta_2)\right), \tag{3.19}$$

$$M_4 = \tau \cdot \left[\left(\psi'''(\beta_1) + \psi'''(\beta_2)\right) + 3\tau \cdot \left(\psi'(\beta_1) + \psi'(\beta_2)\right)^2\right]. \tag{3.20}$$

Proof: According to Lemma 2.1 $E(X) = \mathcal{M}'(0) = \tau(\psi(\beta_1) - \psi(\beta_2))$. Before proofing the other results, it is useful to calculate the first three derivatives of $\mathcal{P}(u)$, which are given by

$$\mathcal{P}'(u) = \tau(\psi'(\beta_1 + u) + \psi'(\beta_2 - u)), \tag{3.21}$$

$$\mathcal{P}''(u) = \tau(\psi''(\beta_1 + u) - \psi''(\beta_2 - u)), \tag{3.22}$$

$$\mathcal{P}'''(u) = \tau(\psi'''(\beta_1 + u) + \psi'''(\beta_2 - u)). \tag{3.23}$$

Furthermore,

$$\begin{aligned} \mathcal{M}''(u) &= \mathcal{M}'(u) \cdot \mathcal{P}(u) + \mathcal{M}(u) \cdot \mathcal{P}'(u) = \mathcal{M}(u) \cdot \mathcal{P}(u)^2 + \mathcal{M}(u) \cdot \mathcal{P}'(u) \\ &= \mathcal{M}(u) \cdot (\mathcal{P}(u)^2 + \mathcal{P}'(u)). \end{aligned}$$

Thus, one can calculate the variance of X by

$$Var(X) = \mathcal{M}''(0) - \mathcal{M}'(0)^2 = \tau \cdot (\psi'(\beta_1) + \psi'(\beta_2)).$$

A further derivative calculation yields

$$\begin{aligned} \mathcal{M}'''(u) &= \mathcal{M}'(u) \cdot (\mathcal{P}(u)^2 + \mathcal{P}'(u)) + \mathcal{M}(u) \cdot (2\mathcal{P}(u)\mathcal{P}'(u) + \mathcal{P}''(u)) \\ &= \mathcal{M}(u)(\mathcal{P}(u)^3 + 3\mathcal{P}'(u)\mathcal{P}(u) + \mathcal{P}''(u)). \end{aligned}$$

Using the relationship between moments and centered moments,

$$E(X - \mu)^3 = \mathcal{M}'''(0) - 3\mathcal{M}''(0)\mathcal{M}'(0) + 2\mathcal{M}'(0)^3 = \mathcal{P}''(0).$$

Finally,

$$\begin{aligned} \mathcal{M}''''(u) &= \mathcal{M}'(u)(\mathcal{P}(u)^3 + 3\mathcal{P}'(u)\mathcal{P}(u) + \mathcal{P}''(u)) + \\ &\quad\ \mathcal{M}(u)(3\mathcal{P}(u)^2\mathcal{P}'(u) + 3\mathcal{P}''(u)\mathcal{P}(u) + 3\mathcal{P}'(u)^2 + \mathcal{P}'''(u)) \\ &= \mathcal{M}(u)(\mathcal{P}(u)^4 + 3\mathcal{P}'(u)\mathcal{P}(u)^2 + \mathcal{P}(u)\mathcal{P}''(u)) + \\ &\quad\ \mathcal{M}(u)(3\mathcal{P}(u)^2\mathcal{P}'(u) + 3\mathcal{P}''(u)\mathcal{P}(u) + 3\mathcal{P}'(u)^2 + \mathcal{P}'''(u)) \\ &= \mathcal{M}(u)\left\{\mathcal{P}(u)^4 + 6\mathcal{P}(u)^2\mathcal{P}'(u) + 4\mathcal{P}(u)\mathcal{P}''(u) + 3\mathcal{P}'(u)^2 + \mathcal{P}'''(u)\right\}. \end{aligned}$$

Applying again relations between moments and centered moments,

$$\begin{aligned} E(X - \mu)^4 &= \mathcal{M}''''(0) - 4\mathcal{M}'''(0)\mathcal{M}'(0) + 6\mathcal{M}''(0)\mathcal{M}'(0)^2 - 3\mathcal{M}'(0)^4 \\ &= 3\mathcal{P}'(0)^2 + \mathcal{P}'''(0). \qquad \Box \end{aligned}$$

Corollary 3.1 (Skewness and kurtosis) *Let X be distributed according to $FEGB2(\beta_1, \beta_2, \tau)$ and $M_i = E(X - m)^i$. Then the skewness and the kurtosis of X are given by*

$$\mathbb{S}(X) = \frac{M_3}{(M_2)^{1.5}} = \frac{1}{\sqrt{\tau}} \cdot \frac{\psi''(\beta_1) - \psi''(\beta_2)}{\sqrt{(\psi'(\beta_1) + \psi'(\beta_2))}^3},$$

$$\mathbb{K}(X) = \frac{M_4}{(M_2)^2} = \frac{(\psi'''(\beta_1) + \psi''(\beta_2)) + 3\tau(\psi'(\beta_1) + \psi'(\beta_2))^2}{\tau \cdot (\psi'(\beta_1) + \psi'(\beta_2))^2}.$$

As we can see in Table 3 – 6 smaller values of τ than 1 allow for skewer and more leptokurtic distributions, whereas higher values of τ than 1 allow for less skew and leptokurtic distributions. In the light of this context it makes sense to restrict τ to the interval $(0, 1]$.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	0	-2.78	-2.83	-2.83	-2.83	-2.83	-2.83	-2.83
0.10	2.78	0	-2.7	-2.74	-2.76	-2.77	-2.77	-2.77
1	2.83	2.7	0	-0.82	-1.31	-1.46	-1.58	-1.6
2	2.83	2.74	0.82	0	-0.62	-0.86	-1.05	-1.08
5	2.83	2.76	1.31	0.62	0	-0.29	-0.58	-0.62
10	2.83	2.77	1.46	0.86	0.29	0	-0.34	-0.4
50	2.83	2.77	1.58	1.05	0.58	0.34	0	-0.08
100	2.83	2.77	1.6	1.08	0.62	0.4	0.08	0

Table 3: Range of skewness for $\tau = 0.5$.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	0	-1.61	-1.63	-1.63	-1.63	-1.63	-1.63	-1.63
0.10	1.61	0	-1.56	-1.58	-1.59	-1.6	-1.6	-1.6
1	1.63	1.56	0	-0.47	-0.75	-0.84	-0.91	-0.92
2	1.63	1.58	0.47	0	-0.36	-0.49	-0.61	-0.62
5	1.63	1.59	0.75	0.36	0	-0.17	-0.33	-0.36
10	1.63	1.6	0.84	0.49	0.17	0	-0.2	-0.23
50	1.63	1.6	0.91	0.61	0.33	0.2	0	-0.05
100	1.63	1.6	0.92	0.62	0.36	0.23	0.05	0

Table 4: Range of skewness for $\tau = 1.5$.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	9	14.76	14.99	14.99	15	15	15	15
0.10	14.76	8.83	14.3	14.52	14.61	14.64	14.66	14.66
1	14.99	14.3	5.4	5.67	6.74	7.24	7.68	7.74
2	14.99	14.52	5.67	4.19	4.37	4.76	5.23	5.3
5	15	14.61	6.74	4.37	3.44	3.45	3.74	3.8
10	15	14.64	7.24	4.76	3.45	3.21	3.3	3.35
50	15	14.66	7.68	5.23	3.74	3.3	3.04	3.04
100	15	14.66	7.74	5.3	3.8	3.35	3.04	3.02

Table 5: Range of kurtosis for $\tau = 0.5$.

$\beta_1 \downarrow \beta_2 \rightarrow$	0.01	0.10	1	2	5	10	50	100
0.01	5	6.92	7	7	7	7	7	7
0.1	6.92	4.94	6.77	6.84	6.87	6.88	6.89	6.89
1	7	6.77	3.8	3.89	4.25	4.41	4.56	4.58
2	7	6.84	3.89	3.4	3.46	3.59	3.74	3.77
5	7	6.87	4.25	3.46	3.15	3.15	3.25	3.27
10	7	6.88	4.41	3.59	3.15	3.07	3.1	3.12
50	7	6.89	4.56	3.74	3.25	3.1	3.01	3.01
100	7	6.89	4.58	3.77	3.27	3.12	3.01	3.01

Table 6: Range of kurtosis for $\tau = 1.5$.

4 Approximations of the FEGB2 density

1. Introduction: Unfortunately, there is no closed form for the density of the FEGB2 distribution, except the case of $\tau = 1$, where it coincides with the EGB2 distribution. Therefore numerical calculations of the probability density function are necessary which are based on the well-known *Fourier-Inversion formula.*

2. Saddlepoint approximation: The following procedure is due to the article of Goutis and Casella [14]. Let us first recall that for a probability density function $f(x)$, the moment generating function is defined as

$$\mathcal{M}(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx,$$

provided that the integral is absolutely integrable. Conversely, one can obtain $f(x)$ from $\mathcal{M}(t)$ by means of the inversion formula (see for example Lukacs [18], page 84):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}(iu) e^{-iux} du = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathcal{K}(iu) - iux} du,$$

where $\varphi(u) = \mathcal{M}(iu)$ denotes the characteristic function and $\mathcal{K}(u) = \log(\mathcal{M}(u))$ the *cumulant generating function* (cgf). Substituting ui by t and using the closed curve theorem from complex analysis we get for ζ close to zero

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\mathcal{K}(t) - tx} dt = \frac{1}{2\pi i} \int_{\zeta - i\infty}^{\zeta + i\infty} e^{\mathcal{K}(t) - tx} dt. \tag{4.24}$$

Expanding the exponent in equation (4.24) around the point $t^* = t^*(x)$ which maximizes the equation $k(t, x) = \mathcal{K}(t) - tx$, i.e. is a solution of the saddlepoint equation $\mathcal{K}'(t) = x$, yields

$$\mathcal{K}(t) - tx \approx \mathcal{K}(t^*) - t^* x + \frac{(t - t^*)^2}{2} \mathcal{K}''(t^*).$$

Plugging this approximation in equation (4.24), integrating with respect to t along the line parallel to the imginary axis through the point t^* finally yields

$$f(x) \approx \frac{\exp(K(t^*) - t^* x)}{\sqrt{2\pi \mathcal{K}''(t^*)}} = \tilde{f}(x). \tag{4.25}$$

Applying the saddlepoint approximation formula (4.25) to the FEGB2 distribution leads to the following result:

Proposition 4.1 (Approximation formula for FEGB2) *Let X be distributed according to FEGB2(β_1, β_2, τ), $\mathcal{M}(u)$ denote the corresponding moment generating function and $\mathcal{P}(u) = \tau[\psi(\beta_1 + u) - \psi(\beta_2 - u)]$. Then*

$$\mathcal{K}(u) = \tau[\ln\Gamma(\beta_1 + u) + \ln\Gamma(\beta_2 - u) - \ln\Gamma(\beta_1) - \ln\Gamma(\beta_2)], \tag{4.26}$$

$$\mathcal{K}'(u) = \mathcal{P}(u). \tag{4.27}$$

Therefore, the density of X can be approximated by

$$f(x) \approx \frac{\exp(\tau\{\ln\Gamma(\beta_1 + u^*) + \ln\Gamma(\beta_2 - u^*) - \ln\Gamma(\beta_1) - \ln\Gamma(\beta_2)\} - u^* x)}{\sqrt{2\pi \mathcal{P}'(u^*)}} \tag{4.28}$$

with $u^* = u^*(x) = \max_{-\beta_1 < u < \beta_2} \{\mathcal{K}(u) - ux\}$.

Proof: (4.26) follows immediately from (3.15). Further

$$\mathcal{K}'(u) = (\ln \mathcal{M}(u))' = \frac{\mathcal{M}'(u)}{\mathcal{M}(u)} = \frac{\mathcal{M}(u)\mathcal{P}(u)}{\mathcal{M}(u)} = \mathcal{P}(u). \quad \square$$

In order to improve the density approximation it is often useful to renormalize the above formula (see for example Ordnung [25]): As the integral over $\tilde{f}$ isn't equal to one, it sometimes turns out to be useful to renormalize the original approximation values by the renormalization contant

$$C = \int_{-\infty}^{\infty} \tilde{f}(x) dx.$$

3. Bohman's approximation: In order to evaluate the integral leading from the characteristic function to the corresponding cumulative distribution function Bohman [4] proposes 5 different methods. The simplest one is given by

$$F(x) \approx \frac{1}{2} + \frac{\vartheta\, x}{2\pi} - \sum_{n=1-N, n\neq 0}^{N-1} \frac{\varphi(\vartheta n)}{2\pi i n} \cdot e^{-i\vartheta n x},$$

where N denotes a positive integer and ϑ a positive quantity. Consequently, differentiation with respect to x yields an approximation formula for the corresponding probability density function:

$$f(x) \approx \frac{\vartheta\, x}{2\pi} + \sum_{n=1-N, n\neq 0}^{N-1} \frac{\vartheta \varphi(\vartheta n)}{2\pi} \cdot e^{-i\vartheta n x}.$$

4. Numerical results: Setting $\tau = 1$ leads to standard EGB2, for which the probability density function is explicitly known. A numerical demonstration of both approximation methods is given in Figure 1 and Figure 2 which show plots of the exact probability density function (EXACT), the saddlepoint approximation (SAP), the renormalized saddlepoint approximation (SAPR) and the Bohman approximation (BOH) for $\mu = 0$, $\delta = 1$, $\beta_1 = 1$, $\beta_2 = 3$ and the corresponding differences. Even though the renormalized saddlepoint approximation comes close to the FEGB-density, the Bohman approximation ($N = 10000$, $\theta = 0.03$) obtains a multiple accuracy.

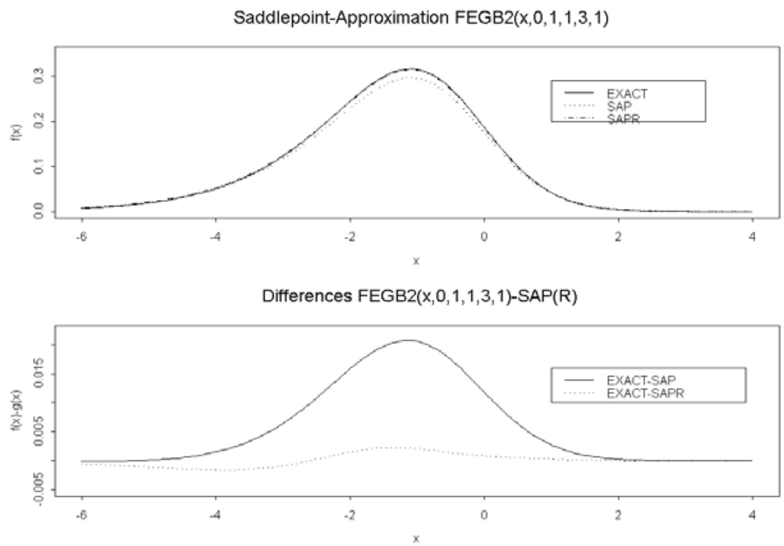

Figure 1: Saddlepoint approximation.

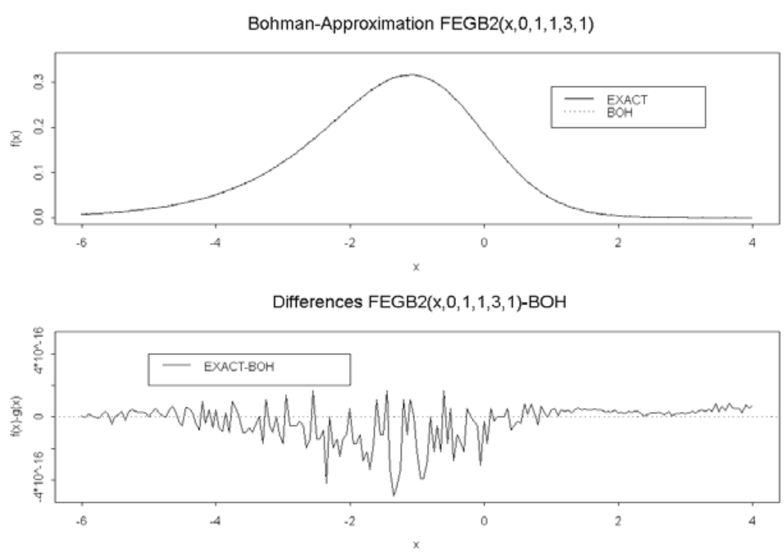

Figure 2: Bohman approximation.

5 Application to finance: Modeling the returns of Nikkei and Mobilcom AG

1. Data: Let us first focus on the weekly returns of the Nikkei 225 from July 31, 1983 to April 9, 1995, with $N = 608$ observations (see Figure 4). This series exhibits typical behaviour of financial return data, that is a considerable kurtosis, to some extend skewness and the presence of volatility cluster (see Figure 3). Results of estimations for various conditional and unconditional distribution models can be found in the work of Mittnik, Paolella and Rachev [24] and be used for comparisons.

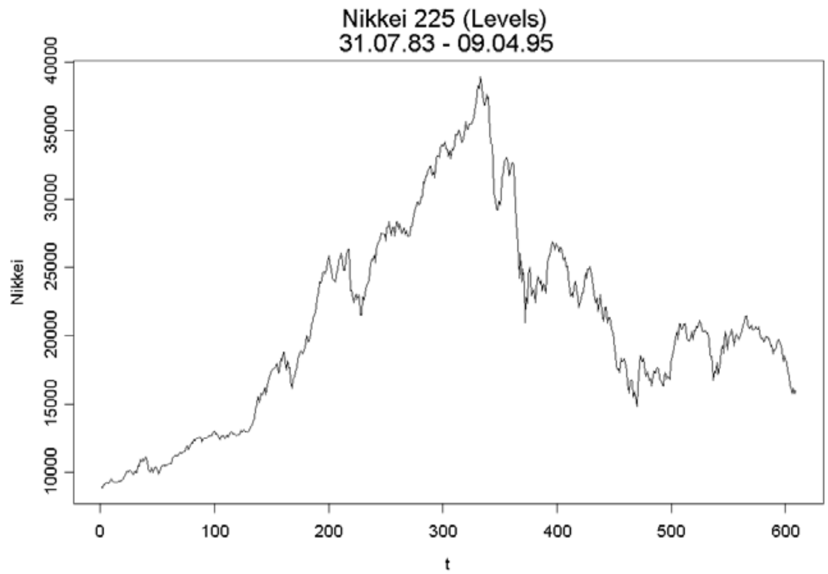

Figure 3: Levels of Nikkei Index.

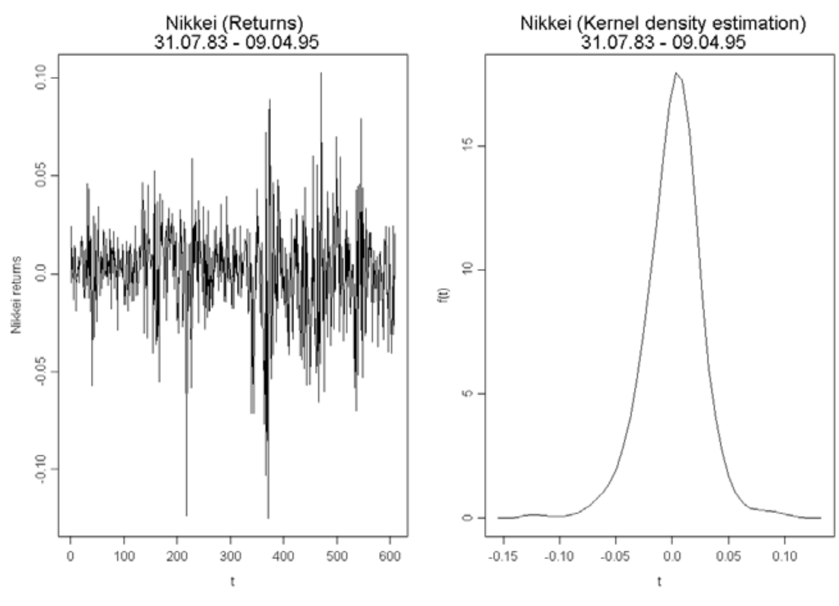

Figure 4: Returns and density estimation of Nikkei Index.

Secondly, let's consider the daily returns of the Neuer Markt participant Mobilcom AG from March 10, 1997 to February 22, 2000, with $N = 743$ observations. Again the levels, returns and density estimation of that series are given in Figure 5 and Figure 6.

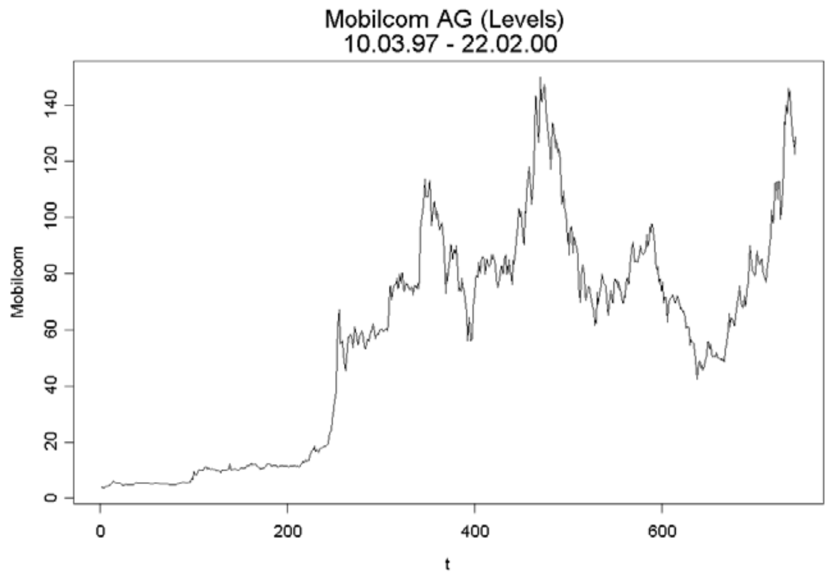

Figure 5: Levels of Mobilcom AG.

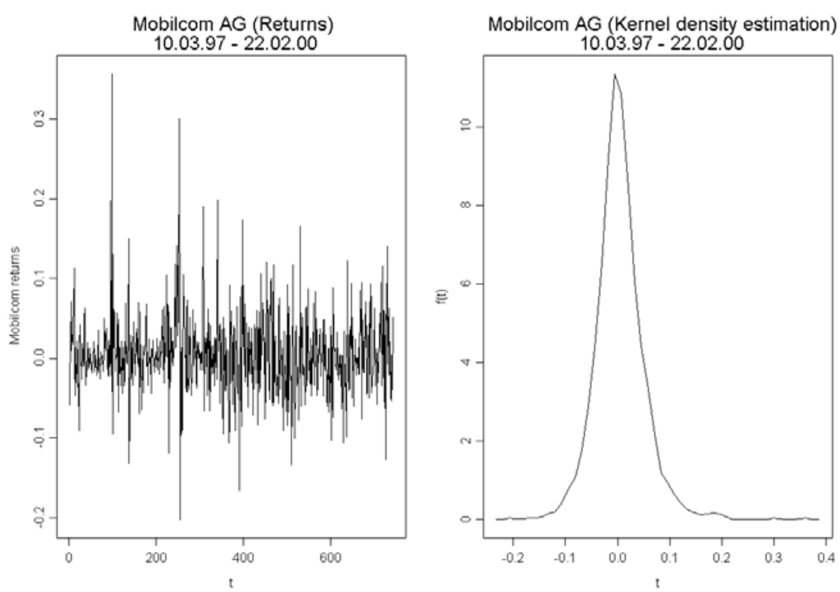

Figure 6: Returns and density estimation of Mobilcom AG.

Ultimately, Table 5 summarizes some statistical properties of the two series (The skewness and kurtosis is measured by the third and fourth standardized moments).

Data	Skewness	Kurtosis
Nikkei	-0.4528	6.16
Mobilcom	1.0848	9.73

Table 5: Skewness and kurtosis.

2. Numerical results of the estimation: Let us start by considering the Nikkei225 data. The Log-Likelihoods (LL) for all members of the logistic family are remarkable higher than those of the normal distribution (see Table 6). However, the additional likelihood increment from EGB2 to FEGB2 is not very remarkable. Obviously, in the case of the Nikkei returns which show kurtosis clearly smaller than nine - i.e. within the range of flexibility of the EGB2 distribution- the additional parameter τ provides no improvement worth mentioning. Calculating the Akaike-Kriterium $AICC = -2LL + \frac{2T(k+1)}{T-k-2}$ even advises to prefer EGB2.

Distrib	μ	δ	β_1	β_2	τ	LL	AICC
Logistic	0.1806	1.3276	**1.0000**	**1.0000**	**1.0000**	-1398.05	2802.14
GL_{III}	0.2495	0.5553	0.3325	0.3325	**1.0000**	-1392.76	2793.59
EGB2	0.6821	0.4525	0.2304	0.3178	**1.0000**	-1388.11	2786.32
FEGB2	0.6594	0.7246	0.3383	0.4718	0.7995	-1388.07	2788.28
Distrib	μ	σ				LL	
Normal	0.0958	2.5349				-1428.26	

Table 6: ML-Estimation parameters for the Nikkei returns.

In order to check the goodness of fit it is useful to calculate some selected distances between the empirical data and the estimated distribution, especially Kolmogorov distance $\mathcal{K}$, the Anderson-Darling distances $\mathcal{AD}_0$, $\mathcal{AD}_1$ and $\mathcal{AD}_2$. The distances corresponding to the distributions above are shown in Table 7:

Distribution	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
Logistic	4.56	0.362	0.236	0.186
GL_{III}	4.35	0.174	0.122	0.121
$GL_{IV}(EGB2)$	2.45	0.103	0.100	0.095
$GL_V(FEGB2)$	2.42	0.095	0.093	0.086
Normal	6.90	4.97	2.83	1.08

Table 7: Goodness of fit for the Nikkei returns.

The same as stated above goes for the $\mathcal{K}$, $\mathcal{AD}_0$, $\mathcal{AD}_1$ and $\mathcal{AD}_2$ values: All distances are smaller in the case of FEGB2, however the decrease being not really mentionable.

Take now the other time series: First of all, the parameters of the ML-estimation of the returns of Mobilcom and the corresponding log-Likelihoods are given in Table 8. Again there is an increase of Log-Likelihood after the introduction of an additional parameter τ.

Distrib	μ	δ	β_1	β_2	τ	LL	AICC
Logistic	0.2637	2.4704	**1.0000**	**1.0000**	**1.0000**	-2171.73	4349.49
GL_{III}	0.1123	0.0501	0.0149	0.0149	**1.0000**	-2158.39	4324.83
EGB2	-0.3820	0.4865	0.1700	0.1321	**1.0000**	-2154.35	4318.77
FEGB2	-0.3809	5.0000	1.2095	0.8310	0.2726	-2153.22	4318.54
Distrib	μ	σ				LL	
Normal	0.4622	4.8739				-2228.11	

Table 8: ML-Estimation parameters for the Mobilcom returns.

Next, the associated distance measures are listed in Table 9. Regarding the results for the returns of Mobilcom, the improvement is now obvious. Moreover AICC for FEGB2 is now smaller than that of EGB2.

Distribution	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
Logistic	3.92	0.821	0.579	0.277
GL_{III}	2.35	0.222	0.144	0.116
$GL_{IV}(EGB2)$	2.33	0.196	0.189	0.069
$GL_V(FEGB2)$	2.17	0.088	0.081	0.046
Normal	8.17	2644.1	103.82	0.87

Table 9: Goodness of fit for the Mobilcom returns.

Across nested models, likelihood ratio tests are suitable to indicate significant differences. For instance, comparing GL_V and GL_{IV} for the Mobilcom data, $\Lambda = -2(-2154.35 - (-2153.22)) = 2.26$. The critical value for $\alpha = 0.05$ yields 3.841, such that H_0: $\tau = 1$ can not be rejected. However, it is highly probable (and gives rise to additional empirical research) that the additional parameter is significant if the underlying data show kurtosis clearly higher than 9.

At last, the fit of the logistic family to both series is shown graphically in Figure 7 and 8. The approximation of FEGB2 was done by Bohman's proposal.

6 Conclusions

As we have seen in the previous pages the introduction of an additional parameter for the EGB2 or generalized logistic function of type V permits the modeling of highly leptokurtic data. Even in the case of "medium" leptocurtosis (that means kurtosis between 3 and 9) there is a slight improvement of fit measured by $\mathcal{K}$, $\mathcal{AD}_0$, $\mathcal{AD}_1$, $\mathcal{AD}_2$ and the corresponding Log-Likelihoods. However, the probability density function of the generalized logistic function of type V is not explicitly known. This causes an additional numerical effort which can be reduced considerably by means of FFT methods.

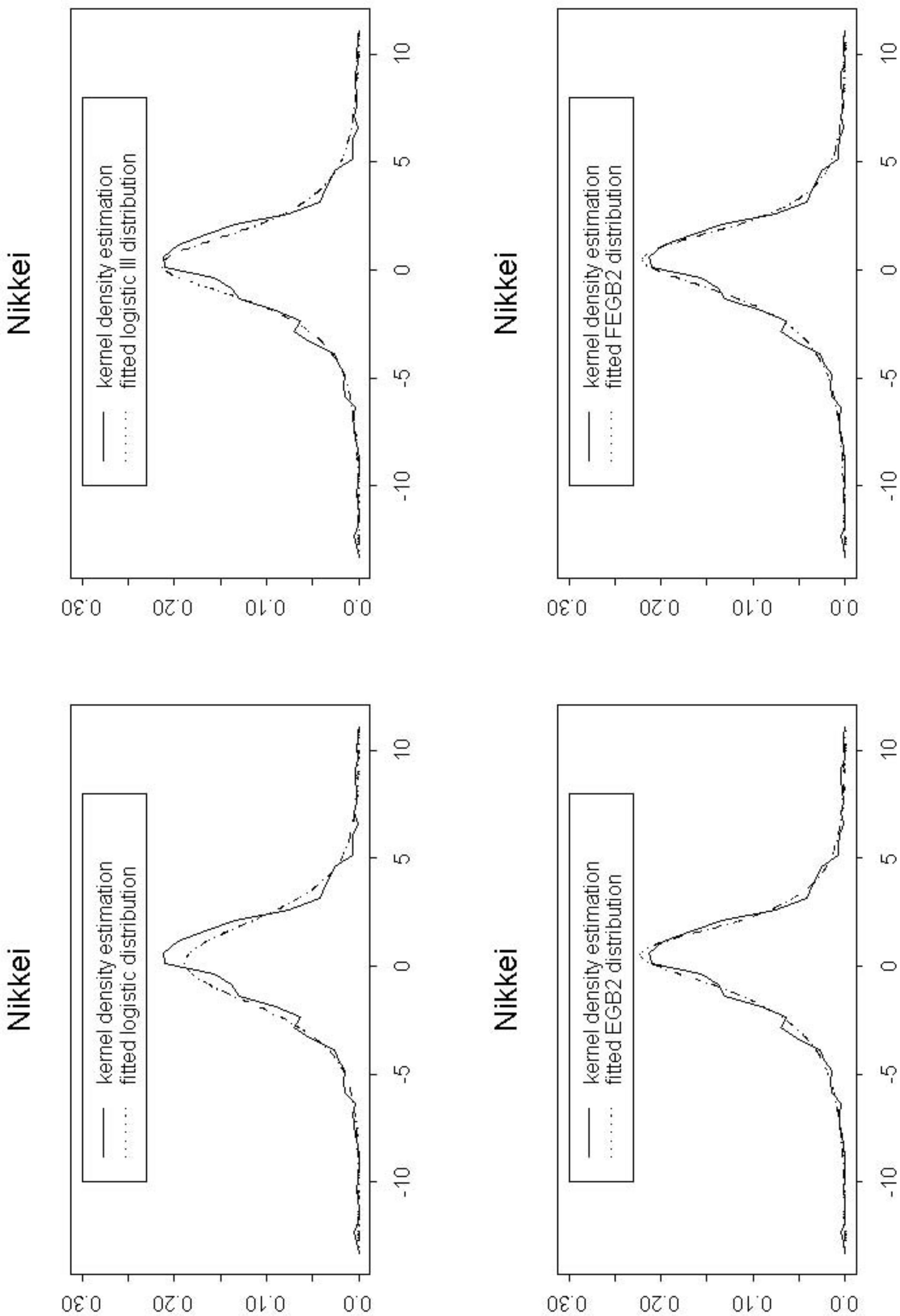

Figure 7: Fitting the Nikkei returns.

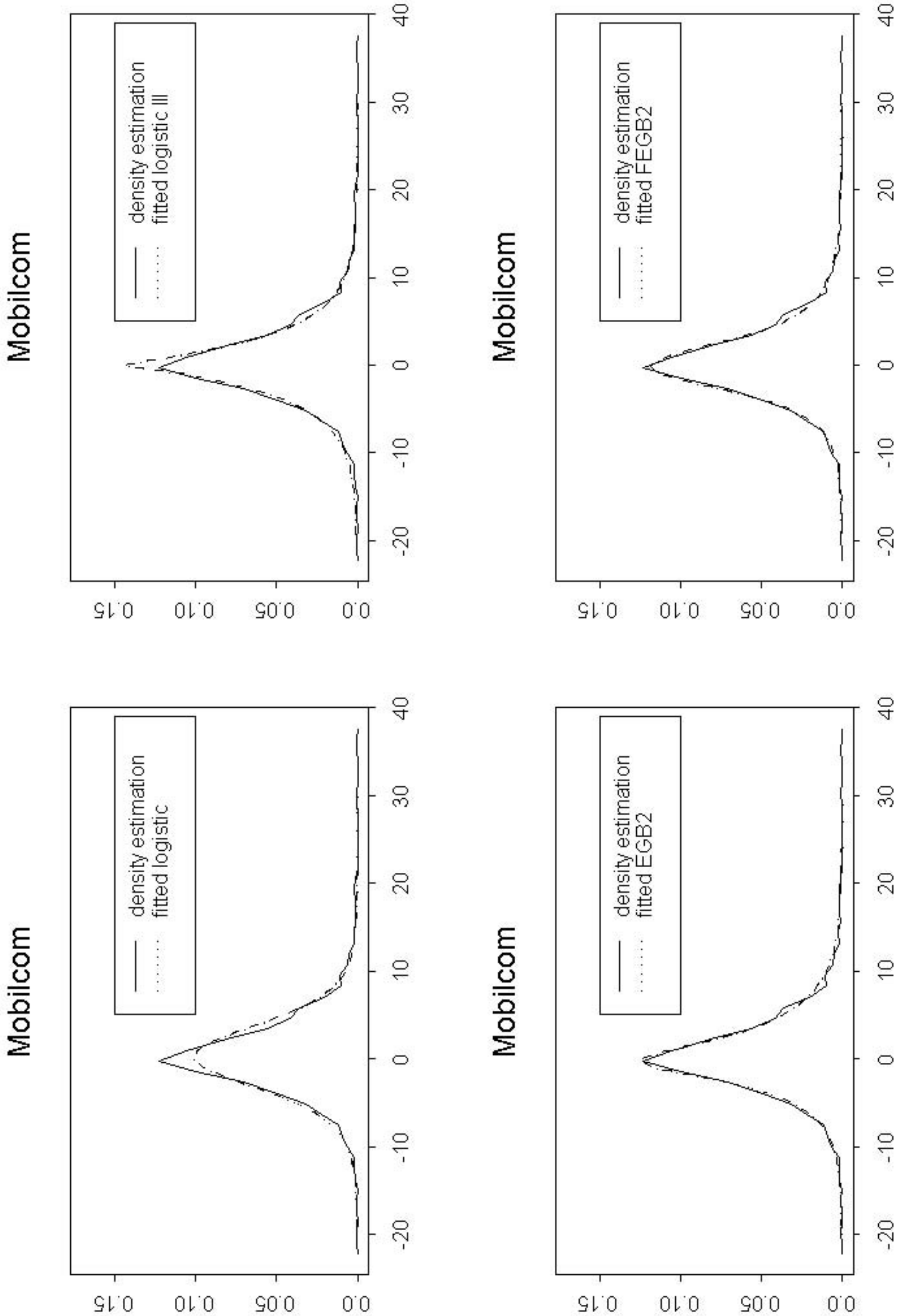

Figure 7: Fitting the Mobilcom returns.

A Beta and Gamma functions

Following Abramowitz and Stegun [2], the Beta function for $x, y \in \mathbb{C}$ with $\mathcal{R}(x) > 0$ and $\mathcal{R}(y) > 0$ is defined by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = 2\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}dt. \tag{1.29}$$

Alternatively, it can be represented in terms of the Gamma function as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{with} \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt. \tag{1.30}$$

Often the derivatives of the logarithm of the Gamma function are very useful, namely Digamma function or Psi function $\psi(x) = \frac{d\ln(\Gamma)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$ and the Polygamma functions $\psi^{(n)}(x) = \frac{d^n\psi(x)}{dx^n} = \frac{d^{n+1}}{dx^{n+1}}\ln(\Gamma(x))$.

B Mixing distributions

The *generalized gamma distribution* (GG) can be defined in terms of the probability density function

$$f_{GG}(x;\gamma,\delta,\beta) = \begin{cases} \frac{|\gamma|}{\delta\,\Gamma(\beta)}\left(\frac{x}{\delta}\right)^{\gamma\beta-1}\exp\left(-\left(\frac{x}{\delta}\right)^{\gamma}\right), & x \geq 0 \\ 0, & x < 0 \end{cases}. \tag{2.31}$$

It was generalized by the *generalized beta of the second kind* (GB2) with probability density function

$$f_{GB2}(x;\alpha,\delta,\beta_1,\beta_2) = \begin{cases} \frac{|\alpha|}{\delta\,B(\beta_1,\beta_2)}\left(\frac{x}{\delta}\right)^{\alpha\beta_1-1}\left(1+\left(\frac{x}{\delta}\right)^{\alpha}\right)^{-(\beta_1+\beta_2)}, & x \geq 0 \\ 0, & x < 0 \end{cases}. \tag{2.32}$$

Besides the *inverse generalized gamma distribution* (IGG) is defined by the probability density function

$$f_{IGG}(x;\gamma,\delta,\beta) = f_{GG}(x;-\gamma,\delta,\beta) = \begin{cases} \frac{|\gamma|}{\delta\,\Gamma(\beta)}\left(\frac{\delta}{x}\right)^{\gamma\beta+1}\exp\left(-\left(\frac{\delta}{x}\right)^{\gamma}\right), & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{2.33}$$

and, using the method of univariate variate transformation, the *exponential generalized gamma distribution* (EGG) with probability density function

$$f_{EGG}(x;\gamma,\delta,\beta) = \frac{|\gamma|}{\delta^{\gamma\beta}\,\Gamma(\beta)}\exp\left(x\gamma\beta - \left(\frac{e^x}{\delta}\right)^{\gamma}\right), \quad x \in \mathbb{R}. \tag{2.34}$$

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