The Esscher-EGB2 option pricing model

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Abstract

With the celebrated model of Black and Scholes in 1973 the development of modern option pricing models started. One of the assumptions of the Black and Scholes model is that the risky asset evolves according to the geometric brownian motion which implies normal distributed returns. As empirical investigations show, the stock returns do not follow a normal distribution, but are leptokurtic and to some extend skewed. The following paper proposes the so-called Esscher-EGB2 option pricing model, where the price process is modeled by an exponential EGB2-Lévy motion, implying that the returns follow an EGB2 distribution and the equivalent martingale measure is given by the Esscher transformation.

1 Preface

The aim of this paper is to provide a more realistic option pricing model than that of Black and Scholes (1973). In the model of Black and Scholes the price process of the underlying stock is modeled by a geometric brownian motion and the equivalent martingale measure is obtained by Girsanov transformation. This implies that the prices of the financial data follow a log-normal distribution or in other words the corresponding returns follow a normal distribution. As various empirical investigations show, however, the returns are not normal distributed, but leptokurtic and to some extend skewed. Section 2 gives a brief description of suitable empirical methods such as the T_3-plot of Gosh (1996) to detect non-normality in financial return data.

Section 3 reviews the definition and some important properties of the so-called generalized logistic distribution of type IV or EGB2 distribution which was already proposed by McDonald (1991) for the description of financial data. This four-parameter distribution is able to model the leptocurtosis and skewness and can be seen as a potential alternative to the hyperbolic distributions or generalized hyperbolic distributions which were suggested by Eberlein et. al. (1995) and Prause (1999), respectively. In contrast to the generalized hyperbolic family, the normalizing constant of the EGB2 density is not determined by the modified Bessel function, but by the Beta function. Estimations could be numerically better tractable in that case.

Finally, in section 4 the EGB2 distribution is used for pricing European options. In order to obtain EGB2-distributed returns one has to choose a more general price process, namely the so-called exponential Lévy processes. So as to arrive at the equivalent martingale measure in this case Gerber and Shiu (1994) picked up the Esscher transformation and applied it to stochastic processes. With the help of Esscher transformation risk-neutral martingale densities could be obtained and used to numerical calculation of option prices. An improvement of speed provides the application of Fast Fourier Transform (FFT). As an alternative to FFT methods we apply the saddlepoint approximation of Rogers and Zane (1999) to the EGB2 option pricing model. This provides a useful tool to compute the probability density function or tail probability by Fourier inversion formula by an expansion of the convex cumulant-generating function at the saddlepoint.

2 Financial return data and normal distribution

Some of the assumptions of the Black and Scholes model are rather debatable. Take for example the compound returns of financial data that means the differences of the logarithms of the prices: To illustrate the non-normality of the these returns let's first consider two graphical tools applied to Deutsche Telekom AG and Mobilcom AG[1] (The levels and the correponding returns of both stocks are visualized in Figure 1), namely the QQ-plot and the T_3-plot of Gosh [10] .

At first, let's focus on the quantile-quantile-plots (QQ-plots) of both shares which are shown in Figure 2. The deviation from a straight line and thus from normality is obvious. It is also evident that there is a considerable mass around the origin and in the tails which can not sufficiently be covered by a normal distribution. Secondly, consider the T_3-plot for Mobilcom AG and Deutsche Telekom AG (Figure 2). Here the third derivative (T_3) of the logarithm of the empirical moment-generating function is proposed as an alternative display. A significant deviation of the T_3-function from the horizontal zero line indicates non-normality; its behaviour in the neighbourhood of 0 indicates the type of departure. The T_3-plot of Mobilcom AG exhibits for example a remarkable skewness and leptocurtosis in the data, whereas the T_3-plot of Deutsche Telekom AG points to the presence of high kurtosis.

Of course, one could also test the assumption of normality with popular test procedures as there are the χ^2 test or the well-known test of Jarque/Bera [13]. These test procedures provide the same results as the graphical tools do.

[1] Mobilcom AG is one of the first participants of the Neuer Markt and shows most of the stylized facts. In contrast to Mobilcom AG, Deutsche Telekom AG is part of the German stock index DAX30. It is also a telecommunication which exhibits moderate behaviour.

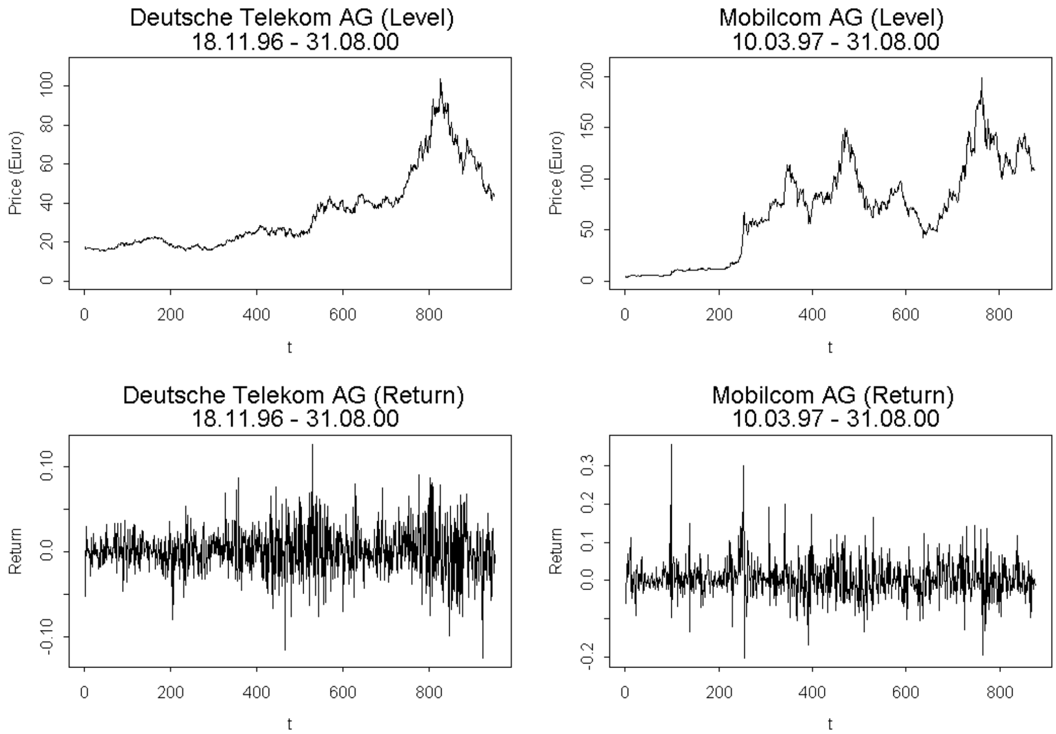

Figure 1: Mobilcom and Deutsche Telekom data.

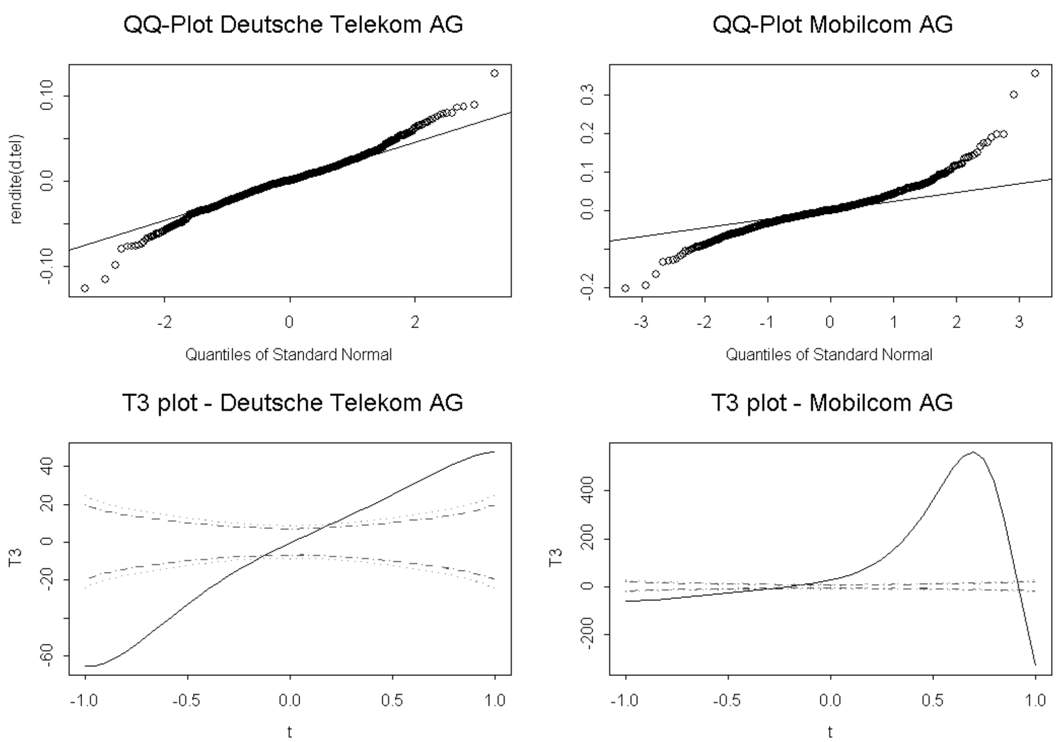

Figure 2: QQ-Plot and T3-Plot.

3 The EGB2 distribution - A review

1. Definition: In the literature there are several generalizations of the logistic distribution. Following the notation of Johnson, Kotz & Balakrishnan [14] three of them are given by the generalized logistic distributions of type I, II and III, briefly denoted by GL_I, GL_{II} and GL_{III} (For a description see Fischer [8]). It is readily observed, that GL_I (setting $\beta_1 = 1, \beta_2 = \alpha$) and GL_{III} (setting $\beta_1 = \beta_2$) are included in the *type IV generalized logistic distribution* (GL_{IV}) with density function

$$f(x; \beta_1, \beta_2) = \frac{1}{B(\beta_1, \beta_2)} \cdot \frac{\exp(\beta_1 x)}{[1 + \exp(x)]^{\beta_1+\beta_2}}, \qquad x \in \mathbb{R}. \tag{3.1}$$

The GL_{IV} is sometimes referred to as the *exponential generalized beta of the second kind*, denoted by EGB2 (McDonald [18]), or as *z-distribution* (Barndorff-Nielsen, Kent, Soerensen [2]). We will follow the notation of McDonald and term this distribution as EGB2 in the sequel.

Introducing a location parameter μ and a scale parameter δ leads to a four parameter family with probability density function

$$f(x; \mu, \delta, \beta_1, \beta_2) = \frac{1}{\delta B(\beta_1, \beta_2)} \cdot \frac{\exp(\beta_1 \frac{x-\mu}{\delta})}{[1 + \exp(\frac{x-\mu}{\delta})]^{\beta_1+\beta_2}}, \qquad x \in \mathbb{R}. \tag{3.2}$$

The positive parameters β_1 and β_2 determine the skewness in the following way:

$$\text{For } \left\{ \begin{array}{c} \beta_1 > \beta_2 \\ \beta_1 < \beta_2 \\ \beta_1 = \beta_2 \end{array} \right\} \text{ the distribution is } \left\{ \begin{array}{l} \text{positively skewed} \\ \text{negatively skewed} \\ \text{symmetric} \end{array} \right\}.$$

Besides we can state the following result for β_1 and β_2:

Lemma 3.1 *β_1 and β_2 are scale- and location-invariant parameters of the EGB2.*

Proof: Let X be EGB2-distributed with parameter vector $\theta = (\mu, \delta, \beta_1, \beta_2)$. It is then easily to show that the distribution of the linear transform $Y = dX + m$ is again EGB2 with parameter vector $\theta^* = (\mu^*, \delta^*, \beta_1^*, \beta_2^*)$=$(m + \mu d, \delta|d|, \beta_1, \beta_2)$ □

2. Special cases: Farewell and Prentice [7] showed that EGB2 goes in limit to lognormal ($\beta_1 \to \infty$), to normal ($\beta_1 \to \infty, \beta_2 \to \infty$) and to Weibull ($\beta_1 = 1, \beta_2 \to \infty$). For $\beta_1 = \beta_2 = 1$ one gets standard logistic distribution, $\text{EGB2}(x; 0, \frac{1}{\sqrt{2\pi}}, \frac{1}{2}, \frac{1}{2})$ coincides with the hyperbolic cosine distribution. Finally, if X is beta distributed with parameter β_1 and β_2 then $\ln(\frac{X}{1-X}) \sim EGB2(x; 0, 1, \beta_1, \beta_2)$.

3. Moment-generating function, moments, skewness and kurtosis: It can be shown that the moment-generating function of X is given by

$$\mathcal{M}_X(t) = \exp(\mu\, t) \cdot \frac{B(\beta_1 + \delta t, \beta_2 - \delta t)}{B(\beta_1, \beta_2)}, \qquad -\frac{\beta_1}{\delta} < t < \frac{\beta_2}{\delta}, \tag{3.3}$$

which leads to the characteristic function

$$\varphi_X(t) = \mathcal{M}(it) = e^{\mu i t} \cdot \frac{B(\beta_1 + i\delta t, \beta_2 - i\delta t)}{B(\beta_1, \beta_2)} = \frac{\Gamma(\beta_1 + i\delta t)\Gamma(\beta_2 - i\delta t)}{\Gamma(\beta_1)\Gamma(\beta_2)}.$$

From (3.3) we get (see also McDonald [18])

$$E(X) = \delta[\psi(\beta_1) - \psi(\beta_2)] + \mu, \tag{3.4}$$

$$M_2 = Var(X) = \delta^2[\psi'(\beta_1) + \psi'(\beta_2)], \tag{3.5}$$

$$M_3 = E[(X-\mu)^3] = \delta^3[\psi''(\beta_1) - \psi''(\beta_2)], \tag{3.6}$$

$$M_4 = E[(X-\mu)^4] = \delta^4\{\psi'''(\beta_1) + \psi'''(\beta_2) + 3[\psi'(\beta_1) + \psi'(\beta_2)]^2\}. \tag{3.7}$$

Hence, one calculate coefficients for skewness and kurtosis measured by the third and fourth standardized moments:

$$\mathbb{S}(X) = \frac{M_3}{M_2^{1.5}} = \frac{\psi''(\beta_1) - \psi''(\beta_2)}{\sqrt{\psi'(\beta_1) + \psi'(\beta_2)}^3}, \tag{3.8}$$

$$\mathbb{K}(X) = \frac{M_4}{M_2^2} = \frac{\psi'''(\beta_1) + \psi'''(\beta_2) + 3[\psi'(\beta_1) + \psi'(\beta_2)]^2}{[\psi'(\beta_1) + \psi'(\beta_2)]^2}. \tag{3.9}$$

It can be shown that the EGB2 can accomodate skewness values between -2 und 2 and positive excess kurtosis up to 6. If the data show clearly higher leptokurtosis it is advisable to use a generalized version of the EGB2, the so-called FEGB2 (see Fischer [8]). The introduction of an additional parameter allows to rebuild arbitrary skewness or leptocurtosis.

4. Self-decomposability and infinite divisibility: Following Barndorff-Nielsen, Kent and Soerensen [2], the EGB2 allows the following normal variance-mean mixture representation[2]: Let $f_m(x;\delta,\gamma)$ be a probability density function of a random variable X on $\mathbb{R}_+$ which has moment generating function

$$\mathcal{M}(t) = \prod_{k=0}^{\infty}\left(1 - \frac{t}{\frac{1}{2}(\delta+k)^2 - \gamma}\right)^{-1}, \qquad \delta > 0, \gamma < \frac{1}{2}\delta^2,$$

i.e. $f_m(x;\delta,\gamma)$ lies in the class of infinite convolution of exponential distribution (Pólya distributions), then EGB2 is a normal variance-mean mixture with mixing distribution f_m. As f_m is a infinite convolution of exponential distributions it belongs to the Thorin class for every (δ,γ). Results of Halgreen [11] and Thorin [21] imply that EGB2 belongs to the extended Thorin class (or class of generalized Gamma convolutions) and are therefore self-decomposable and hence infinitely divisible.

[2]Suppose X is a random variate which is Gauss distributed with mean $\mu + \beta u$ and variance u. Suppose moreover that u itsself follows a probability function F on $\mathbb{R}_+$. Then the distribution of X is said to be a **normal variance-mean mixture** with mixing distribution F. If $\beta = 0$ it is termed a normal variance mixture.

5. ML-Estimation: Supposing $x_1, ..., x_n$ to be i.i.d. returns, the Likelihood function is given by

$$L(\mu, \delta, \beta_1, \beta_2; x_1, .., x_n) = \left(\frac{1}{\delta B(\beta_1, \beta_2)}\right)^n \prod_{i=1}^{n} \frac{\exp(\beta_1 \frac{x_i - \mu}{\delta})}{[1 + \exp(\frac{x_i - \mu}{\delta})]^{\beta_1 + \beta_2}},$$

Therefore the corresponding Log-Likelihood function can be represented as

$$logL(.) = -n \log(\delta B(\beta_1, \beta_2)) + \frac{\beta_1}{\delta} \sum_{i=1}^{n} (x_i - \mu) - (\beta_1 + \beta_2) \sum_{i=1}^{n} \left[\log\left\{1 + e^{\left(\frac{x_i - \mu}{\delta}\right)}\right\}\right].$$

The first partial derivatives of the Log-Likelihood function are given by

$$\frac{\partial \log L}{\partial \mu} = -\frac{\beta_1 n}{\delta} - \frac{\beta_1 + \beta_2}{\delta} \sum_{i=1}^{n} \frac{e^{z_i}}{1 + e^{z_i}},$$

$$\frac{\partial \log L}{\partial \delta} = -\frac{n}{\delta} - \frac{\beta_1 \left(-n\mu + \sum_{i=1}^{n} x_i\right)}{\delta^2} - \frac{(\beta_1 + \beta_2)}{\delta^2} \sum_{i=1}^{n} (x_i - \mu) \frac{e^{z_i}}{1 + e^{z_i}},$$

$$\frac{\partial \log L}{\partial \beta_1} = -n \left[\psi(\beta_1) - \psi(\beta_1 + \beta_2)\right] - \frac{n\mu - \sum_{i=1}^{n} x_i}{\delta} - \sum_{i=1}^{n} \ln(1 + e^{z_i}),$$

$$\frac{\partial \log L}{\partial \beta_2} = -n \left[\psi(\beta_2) - \psi(\beta_1 + \beta_2)\right] - \sum_{i=1}^{n} \ln(1 + e^{z_i}),$$

where $z_i = \frac{x_i - \mu}{\delta}$. The second partial derivatives calculate as

$$\frac{\partial \log L}{\partial \mu \partial \mu} = -\frac{\beta_1 + \beta_2}{\delta^2} \sum_{i=1}^{n} \frac{e^{z_i}}{(1 + e^{z_i})^2},$$

$$\frac{\partial \log L}{\partial \mu \partial \delta} = \frac{\beta_1 n}{\delta^2} - (\beta_1 + \beta_2) \sum_{i=1}^{n} \left\{\frac{e^{z_i}}{\delta^2 (1 + e^{z_i})} + \frac{(x_i - \mu) e^{z_i}}{\delta^3 (1 + e^{z_i})} - \frac{(x_i - \mu) (e^{z_i})^2}{\delta^3 (1 + e^{z_i})^2}\right\},$$

$$\frac{\partial \log L}{\partial \mu \partial \beta_1} = -\frac{1}{\delta} \left(n - \sum_{i=1}^{n} \frac{e^{z_i}}{1 + e^{z_i}}\right),$$

$$\frac{\partial \log L}{\partial \mu \partial \beta_2} = \frac{1}{\delta} \sum_{i=1}^{n} \frac{e^{z_i}}{1 + e^{z_i}},$$

$$\frac{\partial \log L}{\partial \delta \partial \delta} = \frac{n}{\delta^2} + 2 \frac{\beta_1 \left(-n\mu + \sum_{i=1}^{n} x_i\right)}{\delta^3}$$

$$- (\beta_1+\beta_2)\sum_{i=1}^{n}\left\{\frac{2\,(x_i-\mu)\,e^{z_i}}{\delta^3\,(1+e^{z_i})}+\frac{(x_i-\mu)^2\,e^{z_i}}{\delta^4\,(1+e^{z_i})}-\frac{(x_i-\mu)^2\,(e^{z_i})^2}{\delta^4\,(1+e^{z_i})^2}\right\},$$

$$\frac{\partial \log L}{\partial\delta\partial\beta_1}=\frac{1}{\delta^2}\left(n\mu-\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}(-x_i+\mu)\,\frac{e^{z_i}}{1+e^{z_i}}\right),$$

$$\frac{\partial \log L}{\partial\delta\partial\beta_2}=-\frac{1}{\delta^2}\sum_{i=1}^{n}(-x_i+\mu)\,\frac{e^{z_i}}{1+e^{z_i}},$$

$$\frac{\partial \log L}{\partial\beta_1\partial\beta_1}=-n\,(\psi'(\beta_1)-\psi'(\beta_1+\beta_2))\,,$$

$$\frac{\partial \log L}{\partial\beta_1\partial\beta_2}=n\,\psi'(\beta_1+\beta_2),$$

$$\frac{\partial \log L}{\partial\beta_2\partial\beta_2}=-n\,(\psi'(\beta_2)-\psi'(\beta_1+\beta_2))\,.$$

6. Numerical results of the estimation: Let us first consider some statistical properties of the percental returns for the selected data (skewness and kurtosis have been measured by the third and fourth standardized moments). Obviously, both shares exhibit leptocurtosis. Whereas Deutsche Telekom returns do not show remarkable skewness, the returns of Mobilcom do.

Data	**n**	**Mean**	**Std.Dev.**	**Skewness**	**Kurtosis**
Mobilcom AG	877	0.371	4.865	0.921	9.219
Deutsche Telekom AG	952	0.097	2.749	-0.018	4.659

Table 1: Statistical Properties of Returns.

Fitting EGB2 and normal distributions to the empirical returns via Maximum-Likelihood method yields:

EGB2	**n**	μ	δ	β_1	β_2	**Log-Like**
Mobilcom AG	877	-0.3948	0.3838	0.1318	0.1048	-2540.39
Deutsche Telekom AG	952	0.1207	0.7221	0.3950	0.3995	-2281.29
Normal	**n**	μ	σ			**Log-Like**
Mobilcom AG	877	0.3707	4.8622			-2628.37
Deutsche Telekom AG	952	0.097	2.7478			-2310.68

Table 2: Parameter estimations for EGB2 and Normal distribution.

Figure 3 shows a graphical comparison between the normal approximation and the EGB2 approximation for the Mobilcom data. Obviously, EGB2 distributions are suitable to rebuild high peakedness and heavier tails much better than normal distribution do. Finally, Figure 5 and 6 provide the corresponding qq-plots. In case of EGB2 the empirical qq-curves come closer to straight lines which indicate a perfect fit.

As a measure for goodness of fit several distances between the estimated cumulative distribution function F_{est} and the empirical cumulative distribution function F_{emp} have been calculated for both stocks, namely the Kolmogorov distance $\mathcal{K}$ and Anderson & Darling statistics $\mathcal{AD}_0$, $\mathcal{AD}_1$, $\mathcal{AD}_2$:

$$\mathcal{K} = \sup_{x\in\mathbb{R}} |F_{emp}(x) - F_{est}(x)|, \tag{3.10}$$

$$\mathcal{AD}_0 = \sup_{x\in\mathbb{R}} \frac{|F_{emp}(x) - F_{est}(x)|}{\sqrt{F_{est}(x)(1 - F_{est}x)}}. \tag{3.11}$$

Instead of just the maximum discrepancy, it is also meaningful to look at the second and the third largest value, which are denoted as $\mathcal{AD}_1$ and $\mathcal{AD}_2$. The results are the following:

Share	Distr	$\mathcal{K}$	$\mathcal{AD}_0$	$\mathcal{AD}_1$	$\mathcal{AD}_2$
Mobilcom AG	EGB2	2.2296	0.2023	0.1804	0.0652
	Norm	8.1083	2259.3	97.591	0.7849
Telekom AG	EGB2	1.8806	0.0503	0.0472	0.0466
	Norm	4.7735	0.7243	0.6720	0.6132

Table 3: Goodness of fit.

Obviously, the fit of the EGB2 distribution is again much better than that of the normal distribution. All distance measures indicate remarkable improvements by using EGB2.

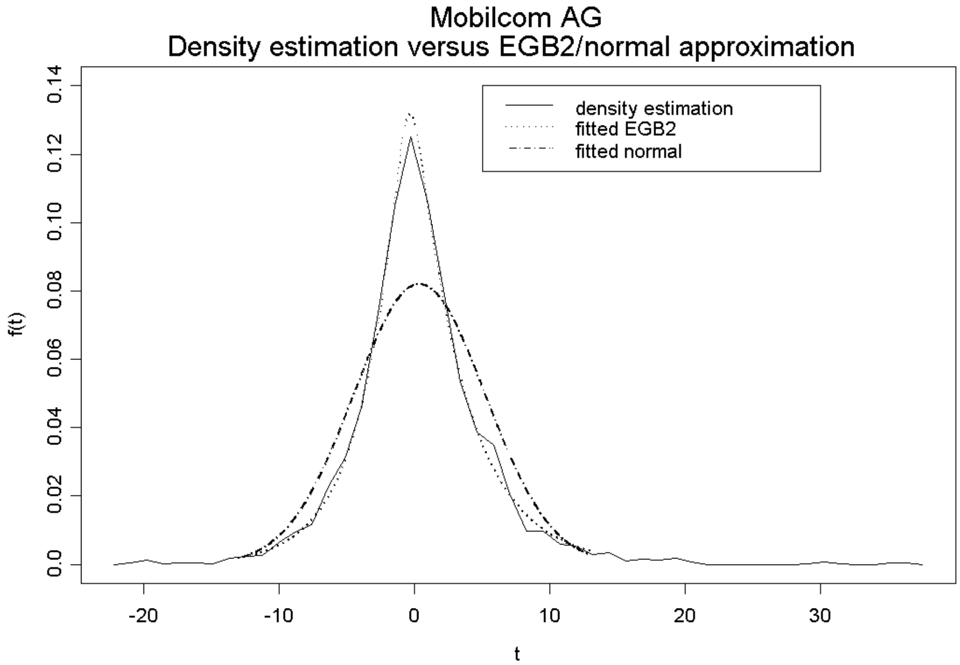

Figure 3: Density estimation (Mobilcom).

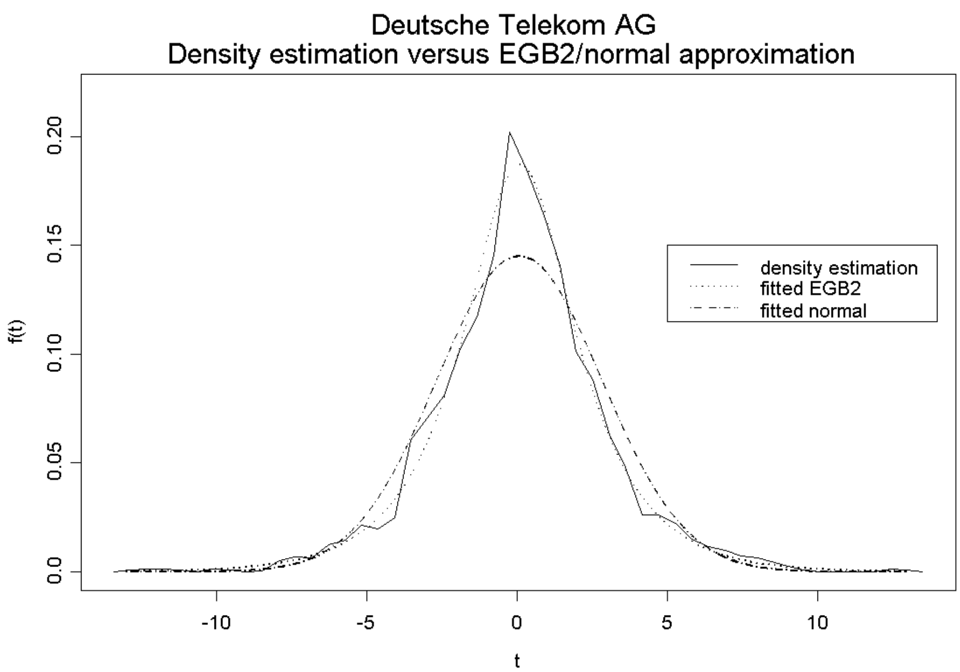

Figure 4: Density estimation (Deutsche Telekom).

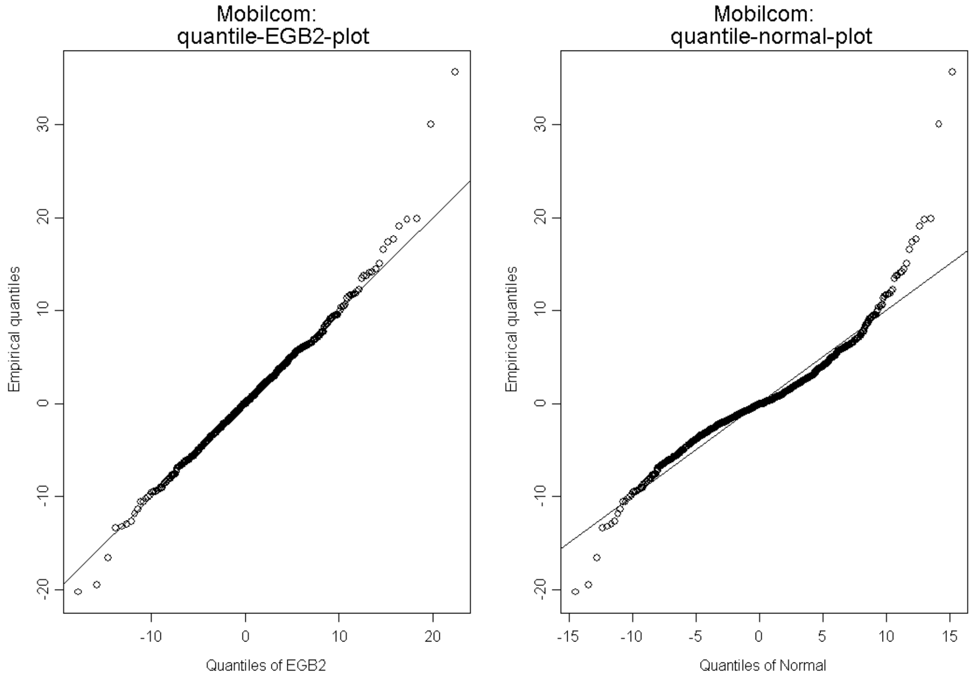

Figure 5: QQ-plots (Mobilcom).

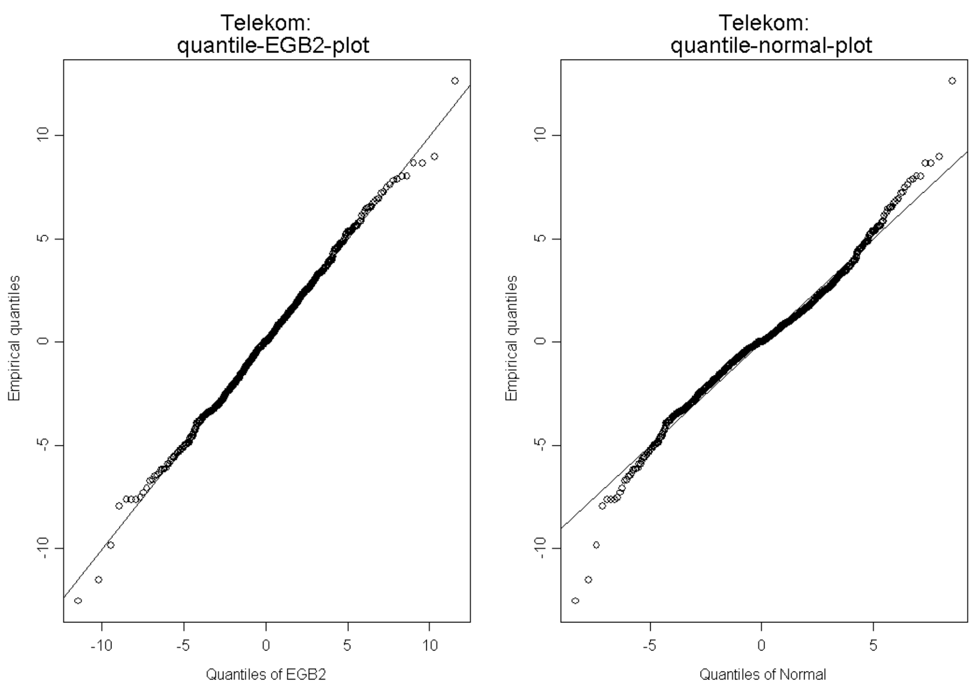

Figure 6: QQ-plots (Deutsche Telekom).

7. Rescaling of EGB2: When fitting a normal distibution only a scale parameter σ and a location parameter μ has to be estimated. In the case of EGB2 distribution additional parameters β_1 and β_2 which determine the skewness and the tails have to be chosen. Tail estimates are typically based on series observed over a longer time horizont, especially rare events like crashes should be taken into account. On the other hand, variance estimates should be adapted regularly with respect to short term developments. The variance of the EGB2-distributed random variable X has a linear structure

$$Var(X) = \delta^2 C_{\beta_1,\beta_2} = \delta^2 \left(\psi'(\beta_1) + \psi'(\beta_2)\right),$$

where C_{β_1,β_2} depends only on the shape, i.e. the scale- and location-invariant parameters β_1, β_2 (see Lemma 2.1). Therefore we use δ as volatility parameter. According to this background, the following rescaling will be executed when pricing an option or calculating implicit volatilities: Given a variance $\hat{\sigma}^2$, the new $\tilde{\delta}$ will be obtained by

$$\tilde{\delta} = \frac{\hat{\sigma}}{\sqrt{\psi'(\beta_1) + \psi'(\beta_2)}}.$$

4 The EGB2 option pricing model

1. Introduction: Option pricing via an equivalent martingale measure: Alternatively to the partial differential equation (PDE) approach the martingale approach is a more general concept which can be applied to other more general models and to other derivatives, for instance options.

The main idea of this approach is based on the no-arbitrage principle which was used by Black and Scholes [3] in order to derive the well-known option pricing formula for European call-options. The central idea of pricing an European option is to construct a hedging portfolio, i.e., a combination of shares from the stock on which the call is written and of shares from a bond market, so that the resulting portfolios replicates the pay-off. At any time, the option should be worth exactly as much as the hedging portfolio, for otherwise some arbitrageurs could make money for nothing (”free lunch”) by trading the option, the stock and the bond market.

A few years later Harrison and Pliska [12] made this concept more explicit. In their terminology the fair value of an option or more general of a contingent claim is given by the *discounted expected value of the pay-off under an eqivalent martingale measure* (EMM). The EMM is sometimes also termed as risk-neutral or risk-adjusted measure. In other words the underlying price process should be a martingale unter the EMM. No arbitrage was shown to be equivalent to the existence of an equivalent martingale measure. Under the assumption of no arbitrage this measure is unique if and only if the market is complete, i.e. every contingent claim can be duplicated by a suitable portfolio.

In the model of Black and Scholes the price process of the underlying stock is modeled by a geometric brownian motion, the equivalent martingale measures is obtained by the Girsanov transformation. This, however, implies the assumption of normal distributed returns which - as we have seen in the previous section - seems not very realistic. In order to obtain generalized distributions for the returns one has to generalize the price process. One way is to assume exponential Lévy processes as a more realistic model for stock prices. This was done for example by Gerber and Shiu [9], Eberlein and Keller [6] or Prause [19]. To derive an equivalent martingale measure we can use the so-called Esscher transformation for example. Originally the concept of the Esscher transform was a time-honored tool in actuarial science. Gerber and Shiu [9] applied this concept to value derivative securities. However, there is of course a price we have to pay: The market is now incomplete. This means it contains non-attainable contingent claims, i.e there are cash flows which cannot be replicated by self-financing trading strategies. If the market is incomplete we have several choices of equivalent martingale measures to price options. In this case it is quite natural to specify the preferences of the agents in order to select one of the martingale measures. The specification of the behaviour of the investor could be done for example in terms of utility functions. Applied to the concept of Esscher transformation we assume an agent with power utiliy function (see Gerber and Shiu [9]).

2. The EGB2-Lévy motion: For every infinitely divisible distribution $\mathcal{L}$ we can easily construct a standardized Lévy process[3] $(X_t)_{t\geq 0}$ such that $X_1 \sim \mathcal{L}$ (see for example Breiman [5]). Since the EGB2 distribution is infinitely divisible there exists a Lévy process $(X_t)t \geq 0$ with $X_1 \sim EGB2$. This process will be termed *EGB2-Lévy motion* in the sequel. Consequently, all increments of length 1 are distributed according to the EGB2 distribution. Let $\mathcal{M}_t$ denote the moment generating function of X_t for any $t > 0$. Using certain properties of Lévy processes the moment generating function can be represented as

$$\begin{aligned} \mathcal{M}_t(u) &= E(\exp(uX_t)) = [E(\exp(uX_1))]^t = M_1(u)^t \\ &= \left(\exp(\mu\, u) \cdot \frac{B(\beta_1 + \delta u, \beta_2 - \delta u)}{B(\beta_1, \beta_2)}\right)^t, \qquad -\frac{\beta_1}{\delta} < u < \frac{\beta_2}{\delta}. \end{aligned} \tag{4.12}$$

As a simple consequence the characteristic function of X_t derives as

$$\varphi_t(u) = M_t(iu) = \left(\exp(\mu\, iu) \cdot \frac{B(\beta_1 + \delta iu, \beta_2 - \delta iu)}{B(\beta_1, \beta_2)}\right)^t. \tag{4.13}$$

Using the *Fourier Inversion formula* one can calculate the probability density function of X_t by means of

$$f_t(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu} \varphi_t(x) dx. \tag{4.14}$$

A plot of the convolution density $f_t(u)$ of the EGB2-Lévy motion for Mobilcom AG is shown in Figure 7 for $t \in [0.2, 2]$ and $u \in [-0.2, 0.2]$. Moments, skewness and kurtosis of f_t are derived in appendix B.

[3] A standardized Lévy process is a stochastic process with stationary and independent increments and starting point $X_0 = 0$.

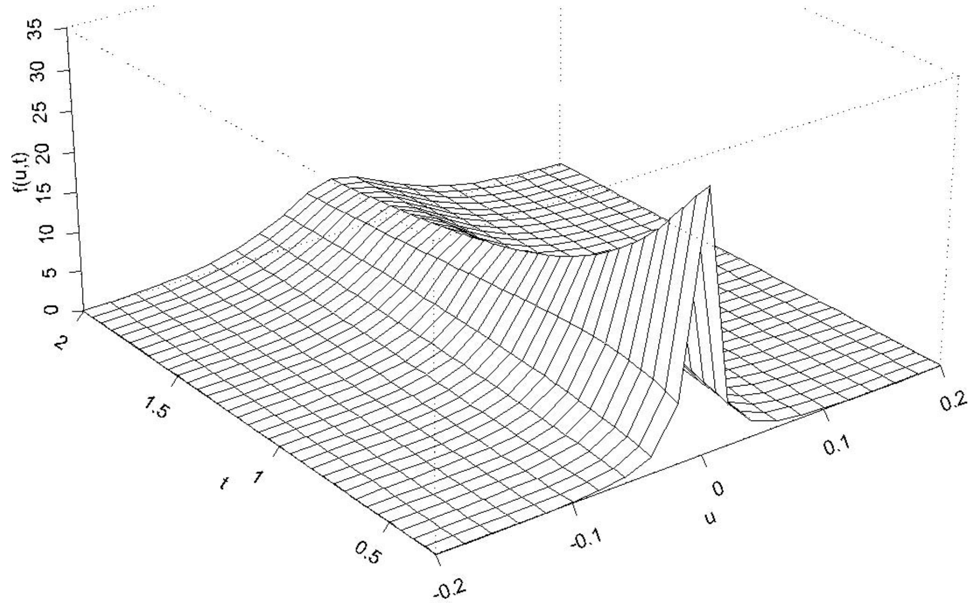

Figure 7: Convolution densities of EGB2-Lévy motion (Mobilcom AG).

3. The exponentiell EGB2-Lévy model: According to the proceeding of Eberlein and Keller ([6]) or Prause ([19]) we consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ which satisfies the usual conditions. We further suppose that there are assets, a risk free asset $(B_t)_{0\leq t\leq T}$ with $B_0 = 1$ that follows the partial differential equation

$$dB_t = rB_t dt, \quad 0 \leq t \leq T.$$

Here r denotes the riskfree rate of interest and T the fixed (finite) time horizont. The second risky asset is a stock whose price process is modeled by an *exponential EGB2 Lévy process* $(S_t)_{t\geq 0}$ with

$$S_t = S_0 e^{X_t}, \quad 0 \leq t \leq T,$$

where $(X_t)_{0\leq t\leq T}$ is an EGB2 Lévy motion.

4. Esscher transformation of the EGB2-Lévy motion: Let $(X_t)_{0\leq t\leq T}$ be an EGB2 Lévy motion and $h \in \mathbb{R}$ for which $\mathcal{M}_t(h)$ exists. Then the *Esscher density* of X_t for the parameter h and $t > 0$ is defined as

$$f_t(x; h) = \frac{e^{hx} f_t(x)}{\int_{-\infty}^{\infty} e^{hy} f_t(y) dy} = \frac{e^{hx} f_t(x)}{\mathcal{M}_t(h)} = \frac{e^{hx}}{\mathcal{M}_1(h)^t} \cdot f_t(x). \tag{4.15}$$

The corresponding characteristic function admits the following representation:

Lemma 4.1 (Characteristic function of the Esscher density) *Let $(X_t)_{t\geq 0}$ be a Lévy process. For any $t>0$ let further $f_t(\cdot)$ denote the corresponding probability density function and $\mathcal{M}_1$ the moment-generating function of X_1. Then, for the associated Esscher transformed Lévy process with density $f_t(\cdot;h)$ the corresponding characteristic function is given by*

$$\varphi_t(u;h) = \frac{\varphi_t(u-ih)}{\mathcal{M}_t(h)} = \left(\frac{\varphi_1(u-ih)}{\mathcal{M}_1(h)}\right)^t. \tag{4.16}$$

Proof: Simple transformations and application of equation (4.15) yields

$$\begin{aligned}\varphi_t(u;h) &= \int_{-\infty}^{\infty} e^{uix} f_t(x,h)dx = \int_{-\infty}^{\infty} e^{uix}\frac{e^{hx}f_t(x)}{\mathcal{M}_t(h)}dx = \frac{1}{\mathcal{M}_t(h)}\int_{-\infty}^{\infty} e^{uix+hx} f_t(x)dx \\ &= \frac{1}{\mathcal{M}_t(h)}\int_{-\infty}^{\infty} e^{(u-hi)ix} f_t(x)dx = \frac{\varphi_t(u-ih)}{\mathcal{M}_t(h)}. \qquad \square\end{aligned}$$

Example 4.1 (CF of the Esscher transformed EGB2-Lévy motion) *Let $(X_t)_{t\geq 0}$ be again an EGB2-Lévy process. Then the characteristic function of the Esscher-transformed density of X_t is given by*

$$\varphi_t(u,h) = \exp(\mu tiu)\cdot\left(\frac{B((\beta_1+\delta h)+\delta iu, (\beta_2-\delta h)-\delta iu)}{B(\beta_1+\delta iu, \beta_2-\delta iu)}\right)^t$$

In terms of Fischer [8] we see that the Esscher-transformed density of X_t follows an $FEGB2$ distribution with parameters $\mu t, \delta, \beta_1+\delta h, \beta_2-\delta h, t$.

Following Gerber and Shiu [9] the process associated to equation (4.16) or (4.15) is again a Lévy process with moment-generating function

$$\mathcal{M}_t(u;h) = \int_{-\infty}^{\infty} e^{ux}\frac{e^{hx}f_t(x)}{\mathcal{M}_t(h)}dx = \frac{\mathcal{M}_t(u+h)}{\mathcal{M}_t(h)} = \mathcal{M}_1(u;h)^t. \tag{4.17}$$

As the exponential function is positive, the correponding Esscher measure is equivalent to the original measure, that means, loosely speaking, both probability measures have the same null sets, i.e. sets with probability measure zero.

Finally, it is desirable to choose a suitable h^* such that the discounted stock price process $e^{-rt}S_t$ is a martingale with respect to the Esscher measure. For these purposes the so-called *martingale equation* can be derived as follows:

$$S_0 = E^*\left(e^{-rt}S_t\right) \iff e^r = \mathcal{M}_1(1,h^*) \iff r = \ln(\mathcal{M}_1(1,h^*)). \tag{4.18}$$

In order to get a martingale one has to choose the Esscher parameter h^* in such a way that the equation on the right hand side of (4.18) is fulfilled. Equivalently, we have to search for a root h^* of the *martingale function*

$$\mathbb{M}(h) = r - \ln(\mathcal{M}_1(1,h)).$$

For instance, the martingale function for the exponential EGB2 Lévy process is given by

$$\mathbb{M}(x;\mu,\delta,\beta_1,\beta_2) = r - \mu + \ln\left(\frac{B(\beta_1+\delta(x+1), \beta_2-\delta(x+1))}{B(\beta_1+\delta x, \beta_2-\delta x)}\right), \quad -\frac{\beta_1}{\delta} < x < \frac{\beta_2}{\delta} - 1.$$

To illustrate this, consider the martingale function in Figure 8 for the EGB2 distribution and the selected financial return data.

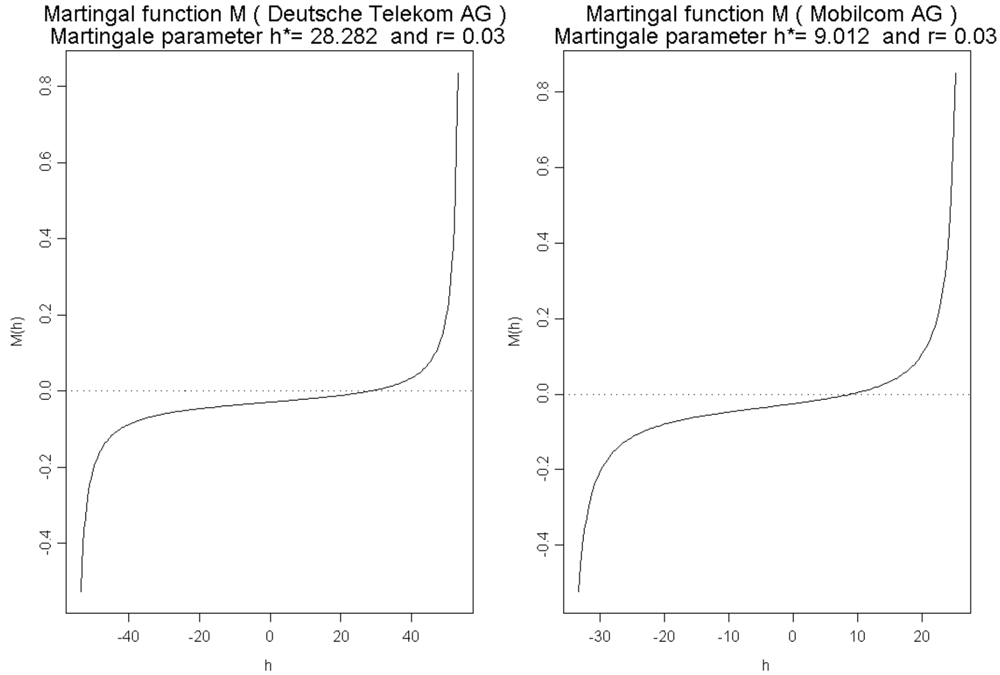

Figure 8: Martingale function (EGB2).

5. Pricing European options via Esscher transformation: As already mentioned in the introductory part about options the fair value C of an European call option on a stock with exercise price B and maturity date T is given by

$$C = e^{-rT} \cdot E^*\Big(\max\{S_T - B; 0\}\Big), \tag{4.19}$$

where E^* denotes the expectation value of a random variable with respect to the equivalent martingale measure, here the risk neutral Esscher measure P^* and r the risk-free interest rate. With $\kappa = \ln(B) - \ln(S_0)$ and

$$e^x \cdot f_T(x; h^*) = e^{rT} \cdot f_T(x; h^* + 1) \tag{4.20}$$

equation (4.19) becomes

$$\begin{aligned} C &= e^{-rT} \int_\kappa^\infty (S_0 e^x - B) f_T(x; h^*) dx \\ &= S_0 \int_\kappa^\infty f_T(x; h^* + 1) dx - e^{-rT} B \int_\kappa^\infty f_T(x; h^*) dx \qquad (4.21) \\ &= S_0 \cdot (1 - F_T(\kappa; h^* + 1)) - e^{-rT} B \cdot (1 - F_T(\kappa; h^*)), \qquad (4.22) \end{aligned}$$

where $F_T(\cdot; h^*)$ denotes the cumulative distribution function of the Esscher transformed Lévy process with characteristic function $\varphi_t(\cdot; h^*)$ given in the previous Lemma 4.1. It can be obtained by means of the Fourier Inversion formula (see for example Kendall, Stuart and Ord [15], chapter 4):

$$F_T(x; h) = \frac{1}{2} - \frac{1}{2\pi} \int_0^\infty \frac{\varphi_T(-u; h) e^{iux} - \varphi_T(u; h) e^{-iux}}{iu} du.$$

6. Numerical implementation and results: To calculate the Esscher prices three major steps have to be run:

1. At first, estimate the parameter of the EGB2 distribution for given return data.

2. Secondly, determine the martingale parameter h^* as the root of martingale function.

3. Finally, the discounted expectation value from (4.19) has to be determined numerically. This could be done in different ways:

   One possibility is to calculate $f_T(\cdot, h^*)$ via (4.16) and (4.15) with the help of Fast Fourier methods and to apply numerical integration methods to evaluate the integrals in (4.21).

   The alternative is to calculate $F_T(\cdot, h^*)$ directly from its corresponding Esscher characteristic function given in Lemma 4.1. Bohman [4], 1975 proposes five different methods to evaluate numerically the integral leading from the characteristic function to the corresponding probability density function, the simplest one is

$$F(x) \approx \frac{1}{2} + \frac{\delta\, x}{2\pi} - \sum_{n=1-N \neq 0}^{N-1} \frac{\varphi(\delta n)}{2\pi i n} \cdot e^{-i\delta n x}.$$

The method described above has been applied to the financial return data of Deutsche Telekom AG. Some results are shown in Figure 9 and 10 where the differences of the Black and Scholes prices minus EGB2 prices are presented. The typical pattern as already observed by Eberlein and Keller [6] is confirmed. For options with a short time to maturity we also state the W-shape, a result of the higher kurtosis of the risk-neutral EGB2 density. Therefore Black-Scholes prices are higher at-the-money and lower in- and out-of-money.

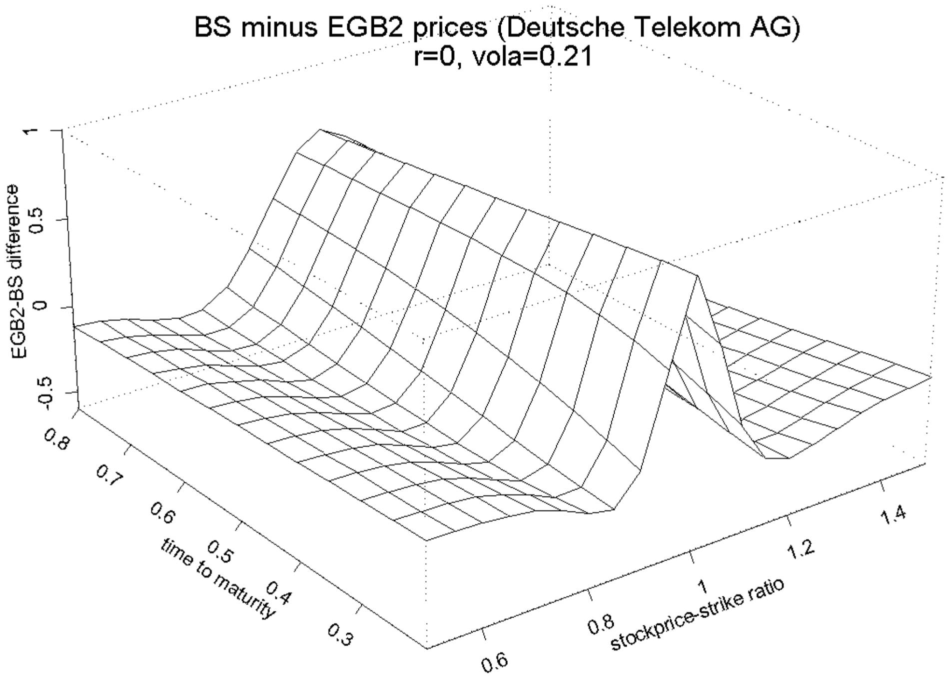

Figure 9: Differences: EGB2 minus BS prices.

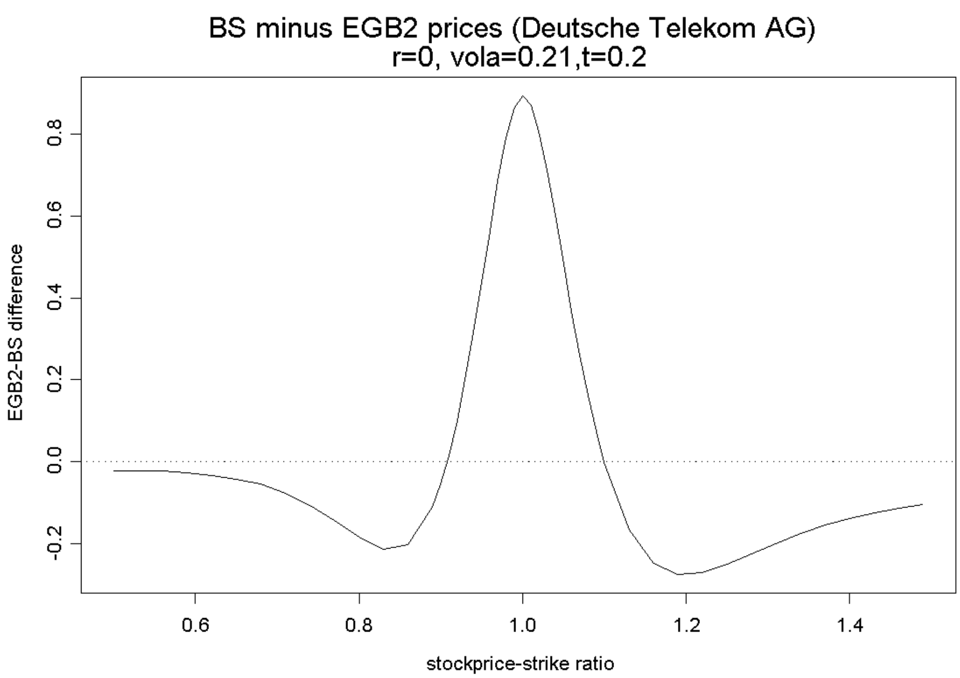

Figure 10: Differences: EGB2 minus BS prices.

7. Saddlepoint Approximation to EGB2-Esscher Prices: As an alternative to FFT methods the saddlepoint approximation provides a useful tool which computes the probability density function or tail probability by Fourier inversion formula by an expansion of the convex cumulant-generating function at the saddlepoint. Hence, the first derivative vanishes and the approximation obtains a simple form. Rogers/Zane [20] proposed the concept of saddlepoint approximation in the context of pricing options. Prause [19] applied the proposal of Roger and Zane to approximate option prices by means of Esscher transformation and generalized hyperbolic distributions. Applied to EGB2-Esscher pricing the proceeding is as follows:

Let $(X_t)_{t\geq 0}$ again denote the EGB2-Lévy process. In particular, the cumulant-generating function of X_T is given by the logarithm of the moment generating function

$$\mathcal{K}_T(u) = \ln(\mathcal{M}_T(u)) = T \cdot \ln(\mathcal{M}_1(u)).$$

Consequently, using relation (4.12) the cumulant-generating function of the Esscher transform is obtained by

$$\begin{aligned}
\mathcal{K}_T(u;h) &= \ln(\mathcal{M}_T(u;h)) = T \cdot (\ln(\mathcal{M}_1(u+h)) - \ln(\mathcal{M}_1(u))) \\
&= T\mu h + \ln\left(\frac{B(\beta_1+\delta(u+h), \beta_2-\delta(u+h))}{B(\beta_1+\delta u, \beta_2-\delta u)}\right) \\
&= T\mu h + \ln\left(\Gamma(\beta_1+\delta(u+h)) + \Gamma\left(\beta_2-\delta(u+h)\right)\right. \\
&\qquad -\Gamma(\beta_1+\delta u) - \Gamma(\beta_2-\delta u).
\end{aligned}$$

Let E^* denote the expectation value with respect to the EMM P^*. Then the fair price of an European put option can be written as[4]

$$\begin{aligned}
p &= E^*\left[e^{-rT}(B-S_T)_+\right] \\
&= S_0 \cdot e^{-rT} \cdot E^*\left[\left(e^{\kappa} - e^{X_T}\right)_+\right] \text{ with } \kappa = \log(B/S_0).
\end{aligned} \tag{4.23}$$

Define the measure P_y^* by a second exponential tilting[5]

$$\frac{dP_y^*}{dP^*} = \exp(yX_T - \mathcal{K}_T(y;h))$$

under the EMM P^*. Using this transformation equation (4.23) changes to

$$p = S_0 \cdot e^{-rT+\kappa} \cdot P^*(X_T < \kappa) - S_0 \cdot e^{-rT+\mathcal{K}_T(1;h)} \cdot P_y^*(X_T < \kappa). \tag{4.24}$$

Besides, the cumulant transform $\mathcal{K}_y(u)$ of P_y^* is given by

$$\mathcal{K}_y(u) = \mathcal{K}_T(y+u;h) - \mathcal{K}_T(y;h).$$

The tail probabilites from equation (4.24) may be approximated by saddlepoint approximation.

[4] The corresponding call value can be obtained via put-call-parity as $c = p + S_0 - e^{-rT}$.

[5] This exponential tilting is not related to the Esscher transform which was used to obtain the equivalent martingale measure.

How to calculate the tail probabilities? One appication of saddlepoint methods is to find approximations of the cumulative distribution function of $\bar{X}_n = \sum_{i=1}^{n} X_i$ where $X_1, ..., X_n$ are i.i.d. random variables with cumulant-generating function $\mathcal{K}(x)$. The following approximation formula goes back to Lugannani and Rice [17] and is valid for all $x \in \mathbb{R}$.

Proposition 4.1 (Lugannani-Rice formula) *Let $x \in \mathbb{R}$. Assume that a solution $\hat{y}(x)$ of the equation $\mu(x) = \mathcal{K}'(x) = x$ exists. Then*

$$P(\bar{X}_n \le x) = \Phi(r(x)) - \varphi(r(x)) \cdot \left(\frac{1}{\lambda(x)} - \frac{1}{r(x)} + O(n^{-3/2}) \right),$$

where φ and Φ are the probability density function and cumulative distribution function of the standard normal distribution, respectively and

$$r(x) = sgn(\hat{y}(x)) \cdot \sqrt{2n \cdot (\hat{y}(x) \cdot x - \mathcal{K}(\hat{y}(x)))},$$

$$\lambda(x) = \sqrt{n} \cdot \hat{y}(x) \cdot \sigma(\hat{y}(x)) \text{ with } \sigma(x) = \mathcal{K}''(x).$$

Corollary 4.1 (Lugannani-Rice for EGB2) *Let us assume that there exist y_1^* and y_2^* satisfying $\mathcal{K}_T'(y_1^*; h) = \kappa$ and $\mathcal{K}_y'(y_2^*) = \kappa$. Applying the above formula to the tail probabilities of equation (4.24) we get*

$$P^*(X_T < \kappa) \approx \Phi(r_1(\kappa)) - \varphi(r_1(\kappa)) \cdot \left(\frac{1}{\lambda_1(\kappa)} - \frac{1}{r_1(\kappa)} \right),$$

$$P_y^*(X_T < \kappa) \approx \Phi(r_2(\kappa)) - \varphi(r_2(\kappa)) \cdot \left(\frac{1}{\lambda_2(\kappa)} - \frac{1}{r_2(\kappa)} \right),$$

with

$$\lambda_i(\kappa) = y_i^* \cdot \sigma_i(y_i^*), \quad i = 1, 2,$$

$$\sigma_1(x) = \mathcal{K}_T''(x; h), \quad \sigma_2(x) = \mathcal{K}_y''(x),$$

$$r_1(x) = sgn(y_1^*) \cdot \sqrt{2 \cdot (y_1^* \cdot \kappa - \mathcal{K}_T(y_1^*; h))},$$

$$r_2(x) = sgn(y_2^*) \cdot \sqrt{2 \cdot (y_2^* \cdot \kappa - \mathcal{K}_y(y_2^*))}.$$

5 Conclusions

The EGB2 option pricing model outlined in this paper provides a more realistic model compared to that of Black and Scholes. Assuming the more flexible EGB2 distribution as model for financial return data leads to modified option prices. Especially, in case of options with short time to maturity Black & Scholes is overpricing at-the-money and underpricing in- and out-of the money. Besides, estimation of EGB2 parameters is particularly easier than estimating parameters of the generalized hyperbolic distribution, whose density includes the modified Bessel function. What still remains to do is to calculate sensitivities such as implicit volatility for the EGB2 model and to compare it with the generalized hyperbolic model.

A Beta and Gamma functions

Following for example Abramowitz and Stegun [1] the Beta function for $x, y \in \mathbb{C}$ with $\mathcal{R}(x) > 0$ and $\mathcal{R}(y) > 0$ is given by the integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = 2\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}dt. \tag{1.25}$$

Alternatively, it can be represented in terms of the Gamma function as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \text{with} \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt. \tag{1.26}$$

Often the derivatives of the logarithm of the Gamma function are very useful, namely Digamma function or Psi function $\psi(x) = \frac{d\ln(\Gamma)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$ and the Polygamma functions $\psi^{(n)}(x) = \frac{d^n\psi(x)}{dx^n} = \frac{d^{n+1}}{dx^{n+1}}\ln(\Gamma(x))$. An algorithm for evaluating arbitrary Polygamma functions was proposed by Klein [16].

B Moments of the EGB2-Lévy motion

Proposition B.1 (Moments) *Let $(X_t)_{t\geq 0}$ denote the EGB2-Lévy motion. Then first four moments of X_t, $t > 0$, are given by*

$$\mu_t = E(X_t) = \mu t + \delta t\left(\psi(\beta_1) - \psi(\beta_2)\right), \tag{2.27}$$

$$M_2 = E(X_t - \mu_t)^2 = t\delta^2 \cdot (\psi'(\beta_1) + \psi'(\beta_2)) = Var(X_t), \tag{2.28}$$

$$M_3 = E(X_t - \mu_t)^3 = t\delta^3 \cdot \left(\psi''(\beta_1) - \psi''(\beta_2)\right), \tag{2.29}$$

$$M_4 = E(X_t - \mu_t)^4 = t\delta^4 \cdot \left[(\psi'''(\beta_1) + \psi'''(\beta_2)) + 3t \cdot (\psi'(\beta_1) + \psi'(\beta_2))^2\right]. \tag{2.30}$$

Proof: As a consequence of infinite divisibility, the moment generating function of X_t is given by

$$\mathcal{M}_t(u) = \left(\exp(\mu\, u) \cdot \frac{B(\beta_1 + \delta u, \beta_2 - \delta u)}{B(\beta_1, \beta_2)}\right)^t.$$

Calculating the first four derivatives yields

$$\begin{aligned}
\mathcal{M}_t'(u) &= \mu t e^{\mu\, ut}\left(\frac{B(\beta_1+\delta u, \beta_2-\delta u)}{B(\beta_1,\beta_2)}\right)^t \\
&\quad + e^{\mu\, ut} t\left(\frac{B(\beta_1+\delta u, \beta_2-\delta u)}{B(\beta_1,\beta_2)}\right)^{t-1} \delta\left(\psi(\beta_1+\delta u) - \psi(\beta_2-\delta u)\right) \\
&= e^{\mu\, ut}\left(\frac{B(\beta_1+\delta u, \beta_2-\delta u)}{B(\beta_1,\beta_2)}\right)^t \Big[\mu t + t\delta\left(\psi(\beta_1+\delta u) - \psi(\beta_2-\delta u)\right)\Big] \\
&= \mathcal{M}_t(u) \cdot \Big[\mu t + t\delta\left(\psi(\beta_1+\delta u) - \psi(\beta_2-\delta u)\right)\Big] = \mathcal{M}_t(u) \cdot \mathcal{P}_t(u),
\end{aligned}$$

implying $E(X_t) = \mathcal{M}_t'(0) = \mu t + \delta t\,(\psi(\beta_1) - \psi(\beta_2))$. Before proofing the other results, it is useful calculating the first three derivatives of $\mathcal{P}_t$, which are given by

$$\mathcal{P}_t'(u) = t\delta^2(\psi'(\beta_1 + \delta u) + \psi'(\beta_2 - \delta u)), \tag{2.31}$$

$$\mathcal{P}_t''(u) = t\delta^3(\psi''(\beta_1 + \delta u) - \psi''(\beta_2 - \delta u)), \tag{2.32}$$

$$\mathcal{P}_t'''(u) = t\delta^4(\psi'''(\beta_1 + \delta u) + \psi'''(\beta_2 - \delta u)). \tag{2.33}$$

Furthermore,

$$\begin{aligned} \mathcal{M}_t''(u) &= \mathcal{M}_t'(u)\cdot\mathcal{P}_t(u) + \mathcal{M}_t(u)\cdot\mathcal{P}_t'(u) = \mathcal{M}_t(u)\cdot\mathcal{P}_t(u)^2 + \mathcal{M}_t(u)\cdot\mathcal{P}_t'(u) \\ &= \mathcal{M}_t(u)\cdot(\mathcal{P}_t(u)^2 + \mathcal{P}_t'(u)). \end{aligned}$$

Thus, one can calculate the variance of X_t by

$$Var(X_t) = \mathcal{M}_t''(0) - \mathcal{M}_t'(0)^2 = t\delta^2\cdot(\psi'(\beta_1) + \psi'(\beta_2))\,.$$

A simple derivative calculation yields

$$\begin{aligned} \mathcal{M}_t'''(u) &= \mathcal{M}_t'(u)\cdot(\mathcal{P}_t(u)^2 + \mathcal{P}_t'(u)) + \mathcal{M}_t(u)\cdot(2\mathcal{P}_t(u)\mathcal{P}_t'(u) + \mathcal{P}_t''(u)) \\ &= \mathcal{M}_t(u)(\mathcal{P}_t(u)^3 + 3\mathcal{P}_t'(u)\mathcal{P}_t(u) + \mathcal{P}_t''(u)). \end{aligned}$$

Using the relationship between moments and centered moments,

$$E(X_t - \mu_t)^3 = \mathcal{M}_t'''(0) - 3\mathcal{M}_t''(0)\mathcal{M}_t'(0) + 2\mathcal{M}_t'(0)^3 = \mathcal{P}_t''(0).$$

Finally,

$$\begin{aligned} \mathcal{M}_t''''(u) &= \mathcal{M}_t'(u)(\mathcal{P}_t(u)^3 + 3\mathcal{P}_t'(u)\mathcal{P}_t(u) + \mathcal{P}_t''(u)) + \\ &\quad \mathcal{M}_t(u)(3\mathcal{P}_t(u)^2\mathcal{P}_t'(u) + 3\mathcal{P}_t''(u)\mathcal{P}_t(u) + 3\mathcal{P}_t'(u)^2 + \mathcal{P}_t'''(u)) \\ &= \mathcal{M}_t(u)(\mathcal{P}_t(u)^4 + 3\mathcal{P}_t'(u)\mathcal{P}_t(u)^2 + \mathcal{P}_t(u)\mathcal{P}_t''(u)) + \\ &\quad \mathcal{M}_t(u)(3\mathcal{P}_t(u)^2\mathcal{P}_t'(u) + 3\mathcal{P}_t''(u)\mathcal{P}_t(u) + 3\mathcal{P}_t'(u)^2 + \mathcal{P}_t'''(u)) \\ &= \mathcal{M}_t(u)\left\{\mathcal{P}_t(u)^4 + 6\mathcal{P}_t(u)^2\mathcal{P}_t'(u) + 4\mathcal{P}_t(u)\mathcal{P}_t''(u) + 3\mathcal{P}_t'(u)^2 + \mathcal{P}_t'''(u)\right\}. \end{aligned}$$

Applying again relations between moments and centered moments,

$$\begin{aligned} E(X_t - \mu_t)^4 &= \mathcal{M}_t''''(0) - 4\mathcal{M}_t'''(0)\mathcal{M}_t'(0) + 6\mathcal{M}_t''(0)\mathcal{M}_t'(0)^2 - 3\mathcal{M}_t'(0)^4 \\ &= 3\mathcal{P}_t'(0)^2 + \mathcal{P}_t'''(0). \qquad \square \end{aligned}$$

Corollary B.1 *Let $(X_t)_{t\geq 0}$ denote the EGB2-Lévy motion. Then the skewness and the kurtosis of X_t, $t > 0$, are given by*

$$\mathbb{S}(X_t) = \frac{M_3}{(M_2)^{1.5}} = \frac{1}{\sqrt{t}}\cdot\frac{\psi''(\beta_1) - \psi''(\beta_2)}{\sqrt{(\psi'(\beta_1) + \psi'(\beta_2))}^3},$$

$$\mathbb{K}(X_t) = \frac{M_4}{(M_2)^2} = \frac{(\psi'''(\beta_1) + \psi''(\beta_2)) + 3t(\psi'(\beta_1) + \psi'(\beta_2))^2}{t\cdot((\psi'(\beta_1) + \psi'(\beta_2)))^2}.$$

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